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Analytic Capacity, Rectifiability, Menger Curvature and the Cauchy Integral

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Introduction

These notes deal with complex analysis, harmonic analysis and geometric measure theory. My main motivation is to explain recent progress on the Painlevé Problem and to describe their connections with the study of the L^2-boundedness of the Cauchy singular integral operator on Ahlfors-regular sets and the quantitative theory of rectifiability.

Let $E \subset \mathbb{C}$ be a compact set. We say that E is **removable** for bounded analytic functions if, for any open set $U \supset E$, any bounded analytic function $f : U \setminus E \to \mathbb{C}$ has an analytic extension to the whole U. The **Painlevé problem** can be stated as follows:

Find a geometric/metric characterization of such removable sets.

In 1947, L. Ahlfors [**1**] introduced the notion of **analytic capacity** of a compact set E:

$$\gamma(E) = \sup\{|f'(\infty)|, f : \mathbb{C} \setminus E \to \mathbb{C} \text{ is analytic bounded with } ||f||_\infty \leq 1\}$$

and proved that E is removable if and only if $\gamma(E) = 0$. But, as wrote Ahlfors himself (in this quotation, $M(G)$ is the analytic capacity of the boundary of G where G is a complex domain of finite connectivity),

"Of course our theorem is only a rather trivial restatement of Painlevé's Problem in what one might call finite terms. But it shows that a "solution" of Painlevé's Problem will be found if we can construct an explicitly defined quantity, depending on G, which tends to zero simultaneously with $M(G)$. Just what is meant by an explicit definition is of course open to discussion, but most mathematicians would probably agree that the ultimate goal is a definition in purely geometric terms. The solution would then be the same general character as one which refers to measure or capacity."

By Riemann's principle for removable singularities, a singleton is removable. On the other hand, by arguments of complex analysis, (non degenerate) continua or compact sets with non zero area are not removable. This suggests that the metric size of the set should play an important role. This observation can be stated more precisely in terms of **Hausdorff dimension** (denoted by $\dim_H$) and 1-dimensional **Hausdorff measure** (denoted by H^1):

(i) If $H^1(E) = 0$, then E is removable.
(ii) If $\dim_H E > 1$, then E is not removable.

Unfortunately, examples of A. Vitushkin, J. Garnett and L. Ivanov [**42**] [**43**] show that the condition $H^1(E) = 0$ is not necessary for the removability of the compact set E. Sets they considered are purely unrectifiable in the sense of geometric measure theory. This leads to the **Vitushkin Conjecture**:

The compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if and only if $Fav(E) = 0$.

Here, $\text{Fav}(E)$ is the Favard length of E and is defined by $\text{Fav}(E) = \int_0^\pi |P_\theta(E)| d\theta$ where P_θ is the projection on the line of $\mathbb{C}$ that makes an angle θ with the real axis and $|P_\theta(E)|$ is the Lebesgue measure of the projection of E on this line. If $H^1(E) < \infty$, the condition $\text{Fav}(E) = 0$ is equivalent to $H^1(E \cap \Gamma) = 0$ for any rectifiable curve Γ of $\mathbb{C}$ (that is E is purely unrectifiable in H. Federer's terminology).
The work of P. Mattila [**63**], P. Jones and T. Murai [**51**] showed that the Vitushkin conjecture is not true for general sets, but left open the case of sets with finite length (that is such that $H^1(E) < \infty$). In 1977, A. P. Calderón [**11**] proved the L^2 boundedness of the **Cauchy operator** on Lipschitz graphs with small constant. This famous result implies a solution to the **Denjoy Conjecture** (and therefore one sense of the Vitushkin conjecture for sets of finite length):

Let $E \subset \mathbb{C}$ be a subset of a rectifiable curve Γ. Then, E is removable if and only if $H^1(E) = 0$.

At that time, it was clear that the removability of a compact set E is closely related to the behavior of the Cauchy operator on E. This motivates the following question:

For which Ahlfors 1-regular sets E is the Cauchy operator bounded on $L^2(E)$ (with respect to the restriction of H^1 to E) ?

A set E in $\mathbb{C}$ is Ahlfors 1-regular if there exists $C > 0$ such that

$$C^{-1}R \leq H^1(E \cap B(x, R)) \leq CR$$

whenever $x \in E$ and $R \in (0, \text{diam}E)$. The example of Lipschitz graphs show that rectifiability properties of the set should play a role. Recall that a set $E \subset \mathbb{C}$ is 1-**rectifiable** if there exist Lipschitz curves Γ_j such that $H^1(E \setminus \cup_j \Gamma_j) = 0$. For this, P. Jones [**50**] (for 1-dimensional sets), G. David and S. Semmes [**28**] [**29**] (in higher dimensions) have developed a quantitative theory of rectifiability.

In 1995, M. Melnikov [**71**] rediscovered the **Menger curvature** and used it to study the semi-additivity of the analytic capacity. The Menger curvature $c(x, y, z)$ of three non collinear points x, y and z of $\mathbb{C}$ is the inverse of the radius of the circumference passing through x, y and z. If the points x, y and z are collinear, we set $c(x, y, z) = 0$. If μ is a positive Radon measure on $\mathbb{C}$, the Menger curvature $c^2(\mu)$ of μ is

$$c^2(\mu) = \int\int\int c(x, y, z)^2 d\mu(x) d\mu(y) d\mu(z).$$

If we assume that $c^2(\mu) < +\infty$, our intuition says that most (with respect to μ) triples are nearly collinear, in other words μ is "flat". In fact, G. David (unpublished) and J.C. Léger [**56**] proved that, if E is a compact set of $\mathbb{C}$ which satisfies $H^1(E) < +\infty$ and $c^2(H^1_{\lceil E}) < +\infty$, then E is 1-rectifiable.

Using the Menger curvature, M. Melnikov and J. Verdera [**72**] gave a simple and geometric proof of the L^2 boundedness of the Cauchy operator on Lipschitz graphs, and with P. Mattila [**69**], they proved that the Cauchy operator is bounded on a

Ahlfors-regular set E if and only if E is contained in a regular curve (that is E is uniformly rectifiable in the sense of G. David and S. Semmes). From this and a previous work of M. Christ [**18**], they proved the Vitushkin conjecture for Ahlfors regular sets. The general case was solved by G. David [**26**].

Very recently, X. Tolsa gave a characterization of removable sets in terms of Menger curvature :

A compact set E of $\mathbb{C}$ is not removable for bounded analytic functions if and only if E supports a positive Radon measure with linear growth and finite Menger curvature.

Recall that a measure μ in $\mathbb{C}$ has linear growth if there exists $C > 0$ such that $\mu(B) \leq C\mathrm{diam}B$ whenever B is a ball in $\mathbb{C}$.

In this book, I would like to tell you this very beautiful story, and I will follow the following plan. In Chapter 1, basic notions of geometric measure theory (like Hausdorff measures, Hausdorff dimension, rectifiable and purely unrectifiable sets) are defined. In particular, we will give several characterizations of rectifiable sets. We will conclude with the proof of a covering lemma by Ahlfors-regular sets. Chapter 2 is devoted to the geometric traveling salesman theorem of P. Jones and the theory of uniformly rectifiable sets of G. David and S. Semmes. In Chapter 3, we will define the Menger curvature and describe some of its properties. In particular, we will show that the Menger curvature is a useful tool to study the geometry of sets and measures in the complex plane. In this part, the reader will find the proofs of some unpublished results of P. Jones. In Chapter 4 is given an overview of the theory of Calderón-Zygmund operators. We also include Melnikov-Verdera's proof of the L^2 boundedness of the Cauchy operator on Lipschitz graphs. The last part of this Chapter will be devoted to the proof of Mattila-Melnikov-Verdera's characterization of Ahlfors-regular sets on which the Cauchy operator is bounded. In Chapter 5, we will define the analytic capacity and we will prove some of its basic properties. The Denjoy and Vitushkin conjectures are proved in Chapter 6. In the last Chapter, we will describe X. Tolsa's characterization of removable sets and we will discuss some open problems.

This book is almost self contained. Only a basic knowledge of real analysis, complex analysis and measure theory is required. Most of the proofs are given. When a proof is omitted or sketched, a reference is indicated where the reader can find a complete proof.

There are good surveys about the subject of this book [**25**] [**67**] [**106**]. I hope that these notes are a complement to these papers and a modest continuation of J. Garnett's Lecture Notes [**42**].

These notes are based on lectures given at the Ecole Normale Supérieure de Lyon and on a graduate course given at Yale University. I would like to thank J. P. Otal and P. Jones for their kind invitation.
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Notations and conventions

If x, $y \in \mathbb{R}^n$, the Euclidean distance between x and y is denoted by $|x-y|$. If $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, $d(x,A) = \inf\{|x-a|; a \in A\}$, and if $B \subset \mathbb{R}^n$, $d(A,B) = \inf\{|a-b|; a \in A, b \in B\}$.
The open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x,r)$. In the special case $n = 2$, that is in $\mathbb{C}$, we will also use the notation $D(x,r)$. For instance, the unit disc in $\mathbb{C}$ is $D(0,1) = \{z \in \mathbb{C}, |z| < 1\}$.
If B is a ball in $\mathbb{R}^n$, we often denote by R_B the radius of B. If $k \in \mathbb{R}^{+*}$, kB is the ball with the same center as B, but whose radius R_{kB} is $k.R_B$.

If E et F are two sets in $\mathbb{R}^n$, then $E + F = \{x + y, x \in E, y \in F\}$ and, for any $x \in \mathbb{R}^n$, $x + F = \{x + y, y \in F\}$.

A measure μ on $\mathbb{R}^n$ for us will be a non-negative, monotonic, subadditive set function which vanishes for empty sets. We always assume that $\mu(\mathbb{R}^n) \neq 0$. A set $A \subset \mathbb{R}^n$ is μ measurable if $\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$ for all $E \subset \mathbb{R}^n$. The measure μ is a Borel measure if all Borel sets are μ measurable. The measure μ is a Radon measure if μ is a Borel measure and satisfies
(i) $\mu(K) < +\infty$ whenever K is a compact set in $\mathbb{R}^n$;
(ii) $\mu(O) = \sup\{\mu(K), K \subset O \text{ compact}\}$ whenever O is an open set in $\mathbb{R}^n$;
(iii) $\mu(A) = \inf\{\mu(O); A \subset O, O \text{ is open }\}$.
If μ is a measure on $\mathbb{R}^n$ and if $E \subset \mathbb{R}^n$, then $\mu_{\restriction E}$ will denote the restriction of μ to E.
The support of a measure μ in $\mathbb{R}^n$ (denoted by Suppμ) is the smallest closed set K such that $\mu(\mathbb{R}^n \setminus K) = 0$.

The Lebesgue measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$.

A dyadic cube Q in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $Q = \Pi_{j=1}^n [k_j 2^{-k}, (k_j+1)2^{-k}]$ where $k \in \mathbb{Z}$ and $k_j \in \mathbb{Z}$. We denote by Δ the set of all dyadic cubes in $\mathbb{R}^n$ and by Δ_k the subset of Δ of k-th generation, that is of side length 2^{-k}. In the special case $n = 1$ (respectively $n = 2$), an element Q of Δ will be called "dyadic interval" (respectively "dyadic square").
Let Q be a dyadic cube in $\mathbb{R}^n$ whose side length is $l(Q)$. Then, if $k \in \mathbb{N}$, kQ denotes the cube with sides parallel to the axis, whose center is the center of Q, but whose side length is $kl(Q)$.

If $E \subset \mathbb{R}^n$, the characteristic function of E is denoted by χ_E.

A constant without a subscript (like C) may vary throughout all the book.

If $A(X)$ and $B(X)$ are two quantities depending on the same variable(s) X, we will say that A and B are comparable if there exists $C \geq 1$ not depending on X such that $C^{-1}A(X) \leq B(X) \leq CA(X)$ for every X.

Contents

CHAPTER 1

Some geometric measure theory

In this chapter, we will define the notions of Hausdorff measures, Hausdorff dimension and rectifiability. Most of the proofs will be omitted. The interested reader is urged to consult [**35**], [**36**] or [**65**].

1. Carleson measures

A measure μ on $\mathbb{R}^n \times \mathbb{R}^+$ is a Carleson measure if there exists $C > 0$ such that

$$\mu(B(x,R) \times (0,R]) \leq CR^n \tag{1}$$

whenever $x \in \mathbb{R}^n$, $R > 0$. The best constant C such that (1) holds is called the Carleson constant of μ. The set $B(x,R) \times (0,R]$ is usually called a Carleson box. The Carleson condition (1) says that the measure of the Carleson box $B(x,R) \times (0,R]$ is controlled by the Lebesgue measure of $B(x,R)$.

A measurable set $A \subset \mathbb{R}^n \times \mathbb{R}^+$ is a Carleson set if $\chi_A(x,t) d\mathcal{L}^n(x) \dfrac{dt}{t}$ is a Carleson measure, that is if there exists $C > 0$ such that

$$\int_{B(x,R)} \int_0^R \chi_A(x,t) d\mathcal{L}^n(x) \frac{dt}{t} \leq CR^n \tag{2}$$

whenever $x \in \mathbb{R}^n$, $R > 0$. The best constant such that (2) holds is the Carleson constant of A.

Example. The set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+; t_0 \leq t \leq 100 t_0\}$ is a Carleson set.

Let $\alpha(x,t)$ be a positive quantity depending on $x \in \mathbb{R}^n$ and on $t \in \mathbb{R}^+$. We will see later several examples of such quantities. Assume that $\alpha(x,t) d\mathcal{L}^n(x) \dfrac{dt}{t}$ is a Carleson measure. Then by Tchebychev inequality, for any $\varepsilon > 0$, the set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \alpha(x,t) > \varepsilon\}$ is a Carleson set (with constant depending on ε).

It should be noticed that a Carleson set is a "small" set in $\mathbb{R}^n \times \mathbb{R}^+$. For instance, let $F \subset \mathbb{N}$ and set $A_F = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+; 2^j < t < 2^{j+1} \text{ for some } j \in F\}$. Then, A_F is a Carleson set if and only if F is finite, and in this case, the Carleson constant is comparable to the number of elements of F (see the nice discussion in [**94**]).

The notion of Carleson measure is one of the main ingredients of L. Carleson's proof of the Corona Theorem [**14**]. More generally, the notion of Carleson measure plays an important role in real and complex analysis. For instance, let $f \in L^p(\mathbb{R})$ (for some $1 < p < +\infty$) and denote by u_f its Poisson integral (see [**44**]). Assume that μ is a measure on $\mathbb{R} \times \mathbb{R}^+$. Then, $u_f \in L^p(\mu)$ for all $f \in L^p(\mathbb{R})$ if and only if μ is a Carleson measure. This result is due to L. Carleson (see [**33**] or [**44**] for a proof and other examples).

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2. Lipschitz maps

A map $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map if there exists $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

whenever x, $y \in E$. The smallest constant K such that this holds is called the Lipschitz constant of f and will be denoted by Lip(f).
A d-dimensional Lipschitz graph is a subset of $\mathbb{R}^n$ of the form $\{(x, f(x)); x \in \mathbb{R}^d\}$ where $f : \mathbb{R}^d \to \mathbb{R}^{n-d}$ is a Lipschitz map or is the image of such a subset by a rotation. Here, we identify $\mathbb{R}^d \times \mathbb{R}^{n-d}$ with $\mathbb{R}^n$.

We now discuss basic properties of Lipschitz maps, namely extension, differentiability and approximation by affine functions.

THEOREM 1 (Kirszbraun's theorem). *Let $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. Then, there exists a Lipschitz map $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(x) = f(x)$ for any $x \in E$ and Lip(g) = Lip(f).*

The proof of this theorem is quite hard (see [**38**] 2.10.43). Instead, we will prove a weaker result. For this, denote by f_k the k-th coordinate function of f. Define $\tilde{g}_k(x) = \inf\{f_k(y) + \text{Lip } (f_k)|x - y|; y \in E\}$ for $x \in \mathbb{R}^n$ and $k = 1, ..., m$. Then, $\tilde{g}_k$ is Lipschitz and Lip$(\tilde{g}_k)$ = Lip(f_k). By using the triangle inequality, you can easily convince yourself that $\tilde{g} = (\tilde{g}_1, ..., \tilde{g}_m)$ is a Lipschitz map and $f = \tilde{g}$ on E. The point is that Lip$(\tilde{g}) \leq \sqrt{n}Lip(f)$ and the equality Lip$(\tilde{g})$ = Lip(f) does not hold in general (see [**35**]).

THEOREM 2 (Rademacher's theorem). *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map, then f is differentiable almost everywhere (with respect to the Lebesgue measure $\mathcal{L}^n$) in $\mathbb{R}^n$.*

We now give a very elegant proof of this result due to A. P. Calderòn. I learned it from [**54**]. See [**35**] or [**65**] for the classical proof.

PROOF. First, note that it is enough to prove the case $m = 1$. Indeed, the general case follows by studying coordinate functions.
In fact, we will prove that every function $u \in W^{1,p}(\mathbb{R}^n)$ with $p > n$ is differentiable almost everywhere. Lipschitz functions correspond to the case $p = \infty$ (see [**54**] theorem 6.12 for more details). Recall that $W^{1,p}(\mathbb{R}^n)$ is the space of L^p integrable functions u such that every weak derivative $\partial_j u$ $(j = 1, ...n)$ is L^p integrable. Since the problem is local, we can assume that u is in $W^{1,p}(B)$ where B is a ball of $\mathbb{R}^n$ (Technical remark: we need this localization in order to use a Sobolev type inequality).
Then, the weak gradient ∇u of u exists almost everywhere in B and almost every point x_0 of B is a Lebesgue point of ∇u, that is

$$\lim_{r\to 0} \fint_{B(x_0,r)} |\nabla u(x) - \nabla u(x_0)|^p d\mathcal{L}^n(x) = 0 \text{ (see [108]).}$$

We claim that if x_0 is a Lebesgue point of ∇u then u is differentiable at x_0 and the differential of u at x_0 is $\nabla u(x_0)$. To see this, set $v(x) = u(x) - u(x_0) - \nabla u(x_0)(x - x_0)$. Then it is clear that $v \in W^{1,p}(B)$ and that $\nabla v(x) = \nabla u(x) - \nabla u(x_0)$ for almost every $x \in B$.
Recall the classical Sobolev embedding theorem (see e.g. [**108**]): If $f \in W^{1,p}(B)$ with $p > n$, then $|f(x) - f(y)| \leq C(n, p)|x - y|^{1-\frac{n}{p}}||\nabla f||_p$ whenever x, y are in B.

We now apply this theorem to v:

$$\begin{aligned}|u(y)-u(x_0)-\nabla u(x_0)(y-x_0)| &= |v(y)-v(x_0)| \\ &\leq C(n,p)|y-x_0|\left(\fint_{B(x_0,|y-x_0|)}|\nabla v(x)|^p d\mathcal{L}^n(x)\right)^{\frac{1}{p}}.\end{aligned}$$

Thus,

$$\frac{|u(y)-u(x_0)-\nabla u(x_0)(y-x_0)|}{|y-x_0|}\leq C(n,p)\left(\fint_{B(x_0,|y-x_0|)}|\nabla u(x)-\nabla u(x_0)|^p d\mathcal{L}^n(x)\right)^{\frac{1}{p}}.$$

Hence, we get

$$\lim_{y\to x_0}\frac{|u(y)-u(x_0)-\nabla u(x_0)(y-x_0)|}{|y-x_0|}=0,$$

and the proof is complete.

□

It is classical in analysis to relate the regularity of a function to some quadratic estimates. A basic example is the following result of E. M. Stein and A. Zygmund (see [**98**] or [**95**]). A function $f:\mathbb{R}\to\mathbb{R}$ is differentiable almost everywhere on a set $E\subset\mathbb{R}$ if and only if for almost every $x\in E$,

$$f(x+t)+f(x-t)-2f(x)=O(|t|)\text{ when }t\to 0;$$

and, for some $\delta>0$,

$$\int_{|t|\leq\delta}\left|\frac{f(x+t)+f(x-t)-2f(x)}{t}\right|^2\frac{dt}{t}<+\infty.$$

In the same spirit, the work of J. Dorronsoro [**31**] gives a more quantitative version of Rademacher's theorem for Lipschitz functions. Before stating his result, we need to introduce some quantities. We follow here the formulation of [**29**].
Given a function $f:\mathbb{R}^n\to\mathbb{R}$ and $1\leq q<\infty$. Define (for all $x\in\mathbb{R}^n$ and $t>0$)

$$\begin{aligned}\gamma_q(x,t) &= \inf t^{-1}\left\{t^{-n}\int_{B(x,t)}|f(y)-a(y)|^q d\mathcal{L}^n(y)\right\}^{\frac{1}{q}}, \\ \gamma_\infty(x,t) &= \inf\left\{t^{-1}\sup_{y\in B(x,t)}|f(y)-a(y)|\right\}\end{aligned}$$

where the infimum is taken over all affine functions $a:\mathbb{R}^n\to\mathbb{R}$.

THEOREM 3. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a Lipschitz function and assume that $1\leq q\leq\infty$ if $n=1$ and $1\leq q<\dfrac{2n}{n-2}$ if $n\geq 2$. Then, there exists $C>0$ such that*

$$\int_0^R\int_{B(x,R)}\gamma_q(y,t)^2 d\mathcal{L}^n(y)\frac{dt}{t}\leq CR^n \tag{3}$$

for all $x\in\mathbb{R}^n$, $t>0$. In other words, $\gamma_q(y,t)^2 d\mathcal{L}^n(y)\dfrac{dt}{t}$ is a Carleson measure on $\mathbb{R}^n\times\mathbb{R}^+$

Roughly speaking, the Carleson estimate (3) allows us to control in each ball the size of the set of (x,t) such that the Lipschitz function f is not well approximated by affine functions at x with respect to the scale t (whereas the Rademacher theorem says that f is infinitesimally well approximated almost everywhere by affine functions). See the discussion of the previous section.

3. Hausdorff dimension and Hausdorff measures

Let $E \subset \mathbb{R}^n$ and let $s > 0$. For $\delta > 0$, consider

$$H^s_\delta(E) = \inf\left\{\sum_i (\text{diam} U_i)^s; E \subset \bigcup_i U_i, \text{diam} U_i \leq \delta\right\}.$$

The s-dimensional Hausdorff measure $H^s(E)$ of E is defined by

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E) = \sup_{\delta > 0} H^s_\delta(E).$$

Then, H^s is a regular Borel measure, but is not a Radon measure if $s < n$ (since H^s is not locally finite if $s < n$).

Assume now that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function with $h(0) = 0$. We define the Hausdorff measure associated to h by

$$\Lambda_h(E) = \lim_{\delta \to 0} \inf\left\{\sum_i h(\text{diam } U_i); E \subset \bigcup_i U_i, \text{diam } U_i \leq \delta\right\}.$$

Therefore, H^s corresponds to the function $h(t) = t^s$.

Proposition 4. *Let $E \subset \mathbb{R}^n$ and let $0 \leq s < t < +\infty$.*

(i) *If $H^s(E) < +\infty$, then $H^t(E) = 0$.*
(ii) *If $H^t(E) > 0$, then $H^s(E) = +\infty$.*

Proof. It is clear that (ii) follows from (i). To prove (i), consider a covering (U_i) of E such that diam $U_i \leq \delta$ for every i, and $\sum_i (\text{diam } U_i)^s \leq H^s_\delta(E) + 1$. Then,

$$H^t_\delta(E) \leq \sum_i (\text{diam } U_i)^t \leq \delta^{t-s} \sum_i (\text{diam } U_i)^s \leq \delta^{t-s}(H^s_\delta(E) + 1).$$

By taking $\delta \to 0$, we get $H^t(E) = 0$. □

We can now define the Hausdorff dimension of $E \subset \mathbb{R}^n$ by

$$\dim_H E = \sup\{s > 0, H^s(E) = +\infty\} = \inf\{t > 0, H^t(E) = 0\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map, $E \subset R^n$ and $0 \leq s < \infty$. Then, $H^s(f(E)) \leq (\text{Lip}(f))^s H^s(E)$. In particular, if p is an orthogonal projection, then $H^s(p(E)) \leq H^s(E)$. See [**35**] for a proof.

Examples:

(a) H^0 is the counting measure, that is $H^0(E) = \text{card} E$ where $\text{card} E$ denotes the number of elements in E .

(b) A curve Γ in $\mathbb{C}$ is a set of the form $\Gamma = \phi([a, b])$ where $[a, b]$ is a closed interval in $\mathbb{R}$ and $\phi : [a, b] \to \mathbb{C}$ is continuous. Moreover, if ϕ is injective, we say that Γ is a

Jordan curve. If ϕ is Lipschitz, we say that Γ is a Lipschitz curve. The length $l(\Gamma)$ of Γ is defined by

$$l(\Gamma) = \sup \sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})|$$

where the supremum is taken over all subdivisions $a = t_0 < t_1 < ..... < t_{n-1} < t_n = b$ of $[a, b]$. If $l(\Gamma) < +\infty$, we say that the curve Γ is rectifiable. It is clear that a Lipschitz curve is rectifiable. If Γ is a rectifiable Jordan curve, define the function $s : [0, l(\Gamma)] \to \mathbb{C}$ by $s(t)$ is the only point $\phi(u)$ such that $l(\phi([a, u])) = t$. Then, s is a parameterization of Γ called the arc length parameterization. Note that s is 1-Lipschitz.

Proposition 5. *If Γ is a Jordan curve in $\mathbb{C}$, then $H^1(\Gamma) = l(\Gamma)$. In particular, if Γ is a rectifiable curve, then $dim_H\Gamma = 1$.*

Proof. We start with a technical result.

Lemma 6. *Let C be a curve joining two points x and y in $\mathbb{C}$. Then, $H^1(C) \geq |x - y|$.*

To prove this lemma, consider p the orthogonal projection on the line passing through x and y. Since p is 1-Lipschitz, we have

$$H^1(C) \geq H^1(p(C)) \geq H^1([x, y]) = \mathcal{L}^1([x, y]) = |x - y|.$$

In the last estimate, we have used the fact that $H^1 = \mathcal{L}^1$ in $\mathbb{R}$ (see below).

We now prove the proposition. Assume that $\Gamma = \phi([a, b])$. Then by the lemma above, for every pair $(u, t) \in [a, b]^2$,

$$H^1(\phi([t, u])) \geq |\phi(u) - \phi(t)|.$$

Therefore, for every subdivision $t_0 = a < t_1 < ..... < t_{n-1} < t_n = b$ of $[a, b]$, we have

$$\sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})| \leq \sum_{i=1}^{n} H^1(\phi([t_{i-1}, t_i])) = H^1(\Gamma).$$

(Note that there is a problem with the endpoints $\phi(t_i)$, but the 1-dimensional Hausdorff measure of a singleton is 0). By taking the supremum over all subdivisions of $[a, b]$, we get $l(\Gamma) \leq H^1(\Gamma)$.

Conversely, if $l(\Gamma) = \infty$, it is obvious. If $l(\Gamma) < \infty$, then Γ is rectifiable. Thus, we can consider the arc length parameterization s of Γ. Since s is 1-Lipschitz, we get

$$H^1(\Gamma) = H^1(s([0, l(\Gamma)])) \leq l(\Gamma).$$

See [**36**] page 29 for a discussion of the case of non Jordan curves.

□

c) If $E \subset \mathbb{C}$, then $H^2(E) = \dfrac{4}{\pi}\mathcal{L}^2(E)$. In particular, $H^2(B(z, r)) = (2r)^2$ whenever $z \in \mathbb{C}$ and $r > 0$. This follows from the isodiametric inequality:

$$\mathcal{L}^2(E) \leq \frac{\pi}{4}(\text{diam}E)^2$$

whenever $E \subset \mathbb{C}$. Note that this inequality is sharp for a ball of diameter diam$E$.

In general, if $E \subset \mathbb{R}^n$, $H^n(E) = c_n \mathcal{L}^n(E)$ and $c_n = \dfrac{2^n (\frac{1}{2}n)!}{\pi^{\frac{n}{2}}}$ (see [**35**]).

In fact, it is not difficult to see that H^n and $\mathcal{L}^n$ are equal up to some constant, that is there exists $c_n > 0$ such that $H^n(E) = c_n \mathcal{L}^n(E)$ if $E \subset \mathbb{R}^n$. This follows from the fact that these two Borel regular measures are uniformly distributed (see [**65**]). The main problem is to give the exact value of c_n and to do this, we need the isodiametric inequality.

d) We now discuss the case of Cantor type sets.
If E is the classical $\dfrac{1}{3}$-Cantor set, then $\dim_H E = \dfrac{\log 2}{\log 3} = s$ and $H^s(E) = 1$ (for a proof, see [**36**]).

We now introduce a planar Cantor set which will play an important role in our story !

Let $E_0 = [0,1]^2$ be the unit cube. Then, E_1 is the union of 4 squares of side length $\dfrac{1}{4}$ and these squares are located in the corners of E_1. In general, E_n is the union of 4^n squares denoted by Q_n^j $(j = 1, ..., 4^n)$ of side length 4^{-n}, each Q_n^j is in the corner of some Q_{n-1}^k. Let $E = \displaystyle\bigcap_{j=0}^{\infty} E_j$.

PROPOSITION 7. *Let E be the four corners Cantor set described above. Then, $0 < H^1(E) < +\infty$.*

PROOF. By Pythagorean theorem, $\mathrm{diam} Q_n^j = \sqrt{2}4^{-n}$. Fix $\delta > 0$. If $4^{-n} < \delta$, then $H^1_\delta(E) \le \sum_{j=1}^{4^n} \mathrm{diam} Q_n^j = \sqrt{2}$. By taking $\delta \to 0$, we get $H^1(E) \le \sqrt{2}$. We now show that $H^1(E) > 0$. Note first that if B is a ball in $\mathbb{C}$ such that diam B $\ge 4^{-n}$ and $B \cap E \ne \emptyset$, then $\sum_{Q_n^j \cap B \ne \emptyset} \mathrm{diam} Q_n^j \le C \mathrm{diam} B$ where C is a constant which does not depend on n. Assume now that $H^1(E) = 0$. Then, we can cover E by balls B_i such that $\sum_i \mathrm{diam} B_i \le \varepsilon$ for some $\varepsilon > 0$. Since E is compact, the collection of balls B_i is finite. Let $\delta = \inf_i \mathrm{diam} B_i$ and let $n \in \mathbb{N}$ such that $\delta > 4^{-n}$. Then, $\sum_i \sum_{Q_n^j \cap B_i \ne \emptyset} \mathrm{diam} Q_n^j \le C\varepsilon$. But, $\sum_i \sum_{Q_n^j \cap B_i \ne \emptyset} \mathrm{diam} Q_n^j$ is comparable to $\sqrt{2}$!!! Hence, $H^1(E) > 0$.

□

Remark. More precise computations show that $H^1(E) = \sqrt{2}$.

We conclude this section with the Frostman lemma which provides a useful tool to estimate Hausdorff dimension.

THEOREM 8. *Let E be a Borel set in $\mathbb{R}^n$. Then, $H^s(E) > 0$ if and only if there exists a finite positive Radon measure μ compactly supported on E such that $\mu(B(x,r)) \le r^s$ for every $x \in \mathbb{R}^n$, $r > 0$.*

See [**65**] for a proof and other versions of this theorem.

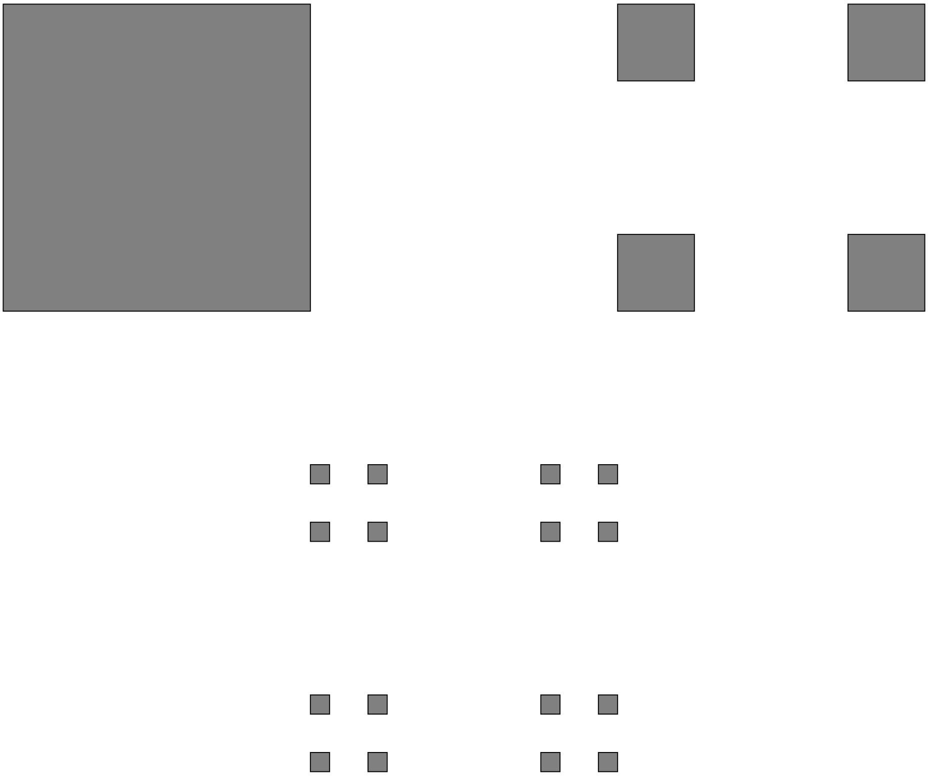

FIGURE 1. The sets E_0, E_1 and E_2

4. Density properties of Hausdorff measures

The classical Lebesgue theorem states that, if $E \subset \mathbb{R}^n$ is a measurable set, $\lim_{r\to 0} \dfrac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))} = 1$ for $\mathcal{L}^n$ almost every $x \in E$ and the limit of the same ratio is equal to 0 for $\mathcal{L}^n$ almost every $x \in \mathbb{R}^n \setminus E$. In this section, we shall see that much less can be said for the Hausdorff measure H^s (if $s \neq n$).

Let $E \subset \mathbb{R}^n$ and let $s \in [0,n]$. We define the lower and upper s-density of E at $x \in \mathbb{R}^n$ by

$$\theta^s_*(x,E) = \liminf_{r\to 0} \frac{H^s(B(x,r)\cap E)}{(2r)^s},$$
$$\theta^{*s}(x,E) = \limsup_{r\to 0} \frac{H^s(B(x,r)\cap E)}{(2r)^s}.$$

If $\theta^s_*(x,E) = \theta^{*s}(x,E)$, we denote by $\theta^s(x,E)$ the common value (called the s-density of E at x).

Remark. Let L be a line in $\mathbb{R}^n$. For every $z \in L$ and $r > 0$, $L \cap B(z,r)$ is an interval that we denote by $I(z,r)$ of diameter $2r$. Thus, for every $z \in L$, $\theta^1(z,L) = 1$ since $H^1(I(z,r)) = 2r$.

PROPOSITION 9. *Let $s \in [0,n]$ and let $E \subset \mathbb{R}^n$ with $H^s(E) < +\infty$.*

(i) $2^{-s} \leq \theta^{*s}(z,E) \leq 1$ *for H^s-almost every $z \in E$;*

(ii) *If E is H^s-measurable, then $\theta^{*s}(z,E) = 0$ for H^s-almost every $z \in \mathbb{R}^n \setminus E$;*

(iii) *If E is H^s-measurable and if F is an H^s-measurable subset of E, then for H^s-almost every $z \in F$,*

$$\theta^s_*(z,F) = \theta^s_*(z,E) \text{ and } \theta^{*s}(z,F) = \theta^{*s}(z,E).$$

For a proof of this proposition, see [**65**].

Remark. There exist compact sets $E \subset \mathbb{R}^n$ with $0 < H^s(E) < \infty$ such that $\theta^s_*(x,E) = 0$ for every $x \in E$ (see [**65**] page 91).

We will use this proposition to prove a covering theorem by Ahlfors-regular sets (see [**86**]).

A set $E \subset \mathbb{R}^n$ is said to be Ahlfors-regular with dimension $d \in \mathbb{R}$ if there exists $C_0 \geq 1$ such that

$$C_0^{-1} r^d \leq H^d(E \cap B(x,r)) \leq C_0 r^d \tag{4}$$

whenever $x \in E$ and $R \in]0, \text{diam}E[$. In this case, $\dim_H E = d$. The best constant such that (4) holds is called the regularity constant of E. Basic examples of Ahlfors regular sets with dimension 1 include lines, Lipschitz graphs and the four corners Cantor set considered in the previous section. Note that for a curve, the lower bound is automatic (at least for small radius). This notion of regularity has been introduced by L. Ahlfors for curves and by G. David for general sets. Ahlfors regular sets are spaces of homogeneous type in the sense of Coifman and Weiss [**19**] which provide a very general setting to do classical analysis (e.g. covering theorems, Lebesgue differentiation theorem, the Calderòn-Zygmund theory of singular integral operators and many other techniques of real analysis work on them). Recall that a metric measure space (X,d,μ) is a space of homogeneous type if the measure μ is doubling: there exists $C_d > 0$ such that $\mu(B(x,2r)) \leq C_d \mu(B(x,r))$ whenever $x \in X$ and $r > 0$. Recently, it has been showed that this doubling condition is not necessary to do analysis in metric spaces (see chapter 6).

Analytic and geometric properties of Ahlfors-regular sets have been widely studied by G. David and S. Semmes [**28**], [**29**].

THEOREM 10. *Let $E \subset \mathbb{R}^n$ be a compact set such that*

(i) $H^d(E) < +\infty$*;*

(ii) $\theta^d_*(x,E) > 0$ *for H^d-almost every $x \in E$.*

Then, $E \subset E_0 \cup (\cup_{k \in \mathbb{N}} F_k)$ where $H^d(E_0) = 0$ and for every $k \in \mathbb{N}$, F_k is Ahlfors regular with dimension d.

We will sketch the proof of this theorem for two reasons. The first one is that this proof involves some classical tools and techniques. The second one is that the original proof is in French !

Proof. Without loss of generality, we can assume that the right hand side of (4) is verified with a constant C_0 for every $x \in E$ and every $r \in (0, \text{diam}E)$.
Indeed,

$$E \subset G_0 \cup (\bigcup_{p\in\mathbb{N}^\star} G_p)$$

where $H^d(G_0) = 0$ and, for every $p \in \mathbb{N}^\star$ (where $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$),

$$G_p = \left\{x \in E : \text{for every } r \in (0, \text{diam}E), H^d(E \cap B(x, r)) \le pr^d\right\}.$$

Moreover, for every $p \in \mathbb{N}^\star$, under the condition $H^d(G_p) \neq 0$,

(i) for any $x \in G_p$, any $r \in (0, \text{diam}E)$,

$$H^d(G_p \cap B(x, r)) \le H^d(E \cap B(x, r)) \le pr^d;$$

(ii) for H^d-almost every $x \in G_p$, $\theta^d_*(x, G_p) = \theta^d_*(x, E) > 0$ (by proposition 9).

Therefore, we can consider G_p instead of E.

The strategy of the proof is clear. First choose a family $(F_{p,s})_{p,s}$ of subsets of E such that $E \subset E_0 \cup (\bigcup_p \bigcup_s F_{p,s})$ where $H^d(E_0) = 0$. Then, for every (p, s), add pieces of d-planes (which are locally d-regular) where the density of $F_{p,s}$ is too small so that the new set $E_{p,s}$ which you obtained by this procedure contains $F_{p,s}$ and is d-regular.

Consider, for every $p \in \mathbb{N}^\star$, for every $s \in \mathbb{N}^\star$, the following sets.

$$\begin{aligned} F_p &= \left\{x \in E : \text{for every } r \in (0, \text{diam}E), H^d(E \cap B(x, r)) > \frac{1}{p} r^d\right\} \\ F_{p,s} &= \left\{x \in F_p : \text{for every } r \in (0, \text{diam}E), H^d(F_p \cap B(x, r)) > \frac{1}{ps} r^d\right\}. \end{aligned}$$

Remark: The choice of $F_{p,s}$ is reasonable. The idea is to consider a "good" covering of $F_p \setminus F_{p,s}$ by balls B and to add to $F_{p,s}$ pieces of d-planes $C_B \subset B$ (of size comparable to $H^d(E \cap B)$) so that the new set obtained $E_{p,s}$ is regular.

We have

(i) For H^d-almost every $x \in E$, $\theta^d_*(x, E) > 0$, and thus there exists $p \in \mathbb{N}^\star$ such that $x \in F_p$.

(ii) For H^d-almost every $x \in F_p$, $\theta^d_*(x, F_p) = \theta^d_*(x, E) > 0$ (see proposition 9).

Thus, (i) and (ii) imply

$$E \subset E_0 \cup (\bigcup_p \bigcup_s F_{p,s}) \tag{5}$$

where $H^d(E_0) = 0$. We shall now construct the d-regular set $E_{p,s}$ which contains $F_{p,s}$.

Recall a classical result (see [**65**], theorem 2.7).

THEOREM 11 (Besicovitch covering theorem). *Let A be a bounded subset of $\mathbb{R}^n$ and let $\mathcal{B}$ be a family of balls such that every point of A is the center of some ball of $\mathcal{B}$.*
Then, there exists a countable collection of balls (B_i) of $\mathcal{B}$ such that

(i) $A \subset \bigcup_i B_i$;

(ii) *every point of $\mathbb{R}^n$ belongs to at most $P(n)$ balls B_i, where $P(n)$ is an integer depending only on n.*

Cover $F_p \setminus F_{p,s}$ by balls $B(y, d(y))$, with $y \in F_p \setminus F_{p,s}$, $d(y) = \frac{1}{10}\text{dist}(y, F_{p,s})$. Then, by theorem 11, there exists a subset $H_{p,s}$ of $F_p \setminus F_{p,s}$ such that $F_p \setminus F_{p,s} \subset \cup_{y \in H_{p,s}} B(y, d(y))$ and any point of $\mathbb{R}^n$ belongs to at most $P(n)$ balls $B(y, d(y))$, $y \in H_{p,s}$. For every $y \in H_{p,s}$, we can construct a piece of d-plane C_y, which is (locally) d-regular, such that

(1) $C_y \subset B(y, d(y))$;
(2) $\theta^{-1}(d(y))^d \le H^d(C_y) \le \theta(d(y))^d$

(for some positive constant $\theta > 0$). Set $E_{p,s} = F_{p,s} \cup (\bigcup_{y \in H_{p,s}} C_y)$.

Then, by (5), $E \subset E_0 \cup (\bigcup_p \bigcup_s E_{p,s})$ (where $H^d(E_0) = 0$) and we claim that, for every $p \in \mathbb{N}^\star$, every $s \in \mathbb{N}^\star$, $E_{p,s}$ is d-regular. We shall only prove the regularity for $x \in F_{p,s}$. For this, fix $p > 0$, $s > 0$ and consider $x \in F_{p,s}$, $r \in (0, \text{diam}E)$. Let us begin with a trivial result (we omit the proof).

LEMMA 12. *Let $y \in H_{p,s}$ such that $B(y, d(y)) \cap B(x, r) \neq \emptyset$.*
Then $d(y) < r$ and $B(y, d(y)) \subset B(x, 10r)$.

We first prove the upper regularity.
By definition of $E_{p,s}$, we have

$$H^d(E_{p,s} \cap B(x,r)) \le H^d(F_{p,s} \cap B(x,r)) + \sum_{\{y \in H_{p,s} : C_y \cap B(x,r) \neq \emptyset\}} H^d(C_y). \tag{6}$$

Because of the upper regularity of E,

$$H^d(F_{p,s} \cap B(x,r)) \le C_0 r^d. \tag{7}$$

Moreover,

$$\begin{aligned}
\sum_{\{y \in H_{p,s} : C_y \cap B(x,r) \neq \emptyset\}} H^d(C_y) &\le \sum_{\{y \in H_{p,s} : B(y,d(y)) \cap B(x,r) \neq \emptyset\}} H^d(C_y) \\
&\le \theta \sum_y (d(y))^d \text{ (by construction of } C_y) \\
&\le \theta \sum_y p H^d(E \cap B(y, d(y))) \text{ (because } y \in F_p) \\
&\le C p \theta P(n) H^d(E \cap B(x, 10r)) \text{ (by lemma 12)} \\
&\le C p \theta P(n) C_0 10^d r^d \qquad (8)
\end{aligned}$$

where all the sums are taken over the set $\{y \in H_{p,s} : B(y, d(y)) \cap B(x,r) \neq \emptyset\}$ and where $P(n)$ is the constant of the Besicovitch covering theorem.

Thus, inequalities (6), (7) and (8) give the upper regularity in x.

We now prove the lower regularity.

By lemma 12 (applied with radius $\frac{r}{10}$),

$$H^d(E_{p,s}\cap B(x,r)) \geq H^d(F_{p,s}\cap B(x,r)) + \sum_{\{y\in H_{p,s}:B(y,d(y))\cap B(x,\frac{r}{10})\neq\emptyset\}} H^d(C_y).$$

Moreover,

$$\begin{aligned}\sum_{\{y\in H_{p,s}:B(y,d(y))\cap B(x,\frac{r}{10})\neq\emptyset\}} H^d(C_y) &\geq \theta^{-1}\sum_y (d(y))^d (\text{ by construction})\\ &\geq \theta^{-1}C_0^{-1}\sum_y H^d(E\cap B(y,d(y)))\\ &\geq \theta^{-1}C_0^{-1}H^d((F_p\setminus F_{p,s})\cap B(x,\frac{r}{10}))\end{aligned}$$

where the sums are taken over the set $\{y\in H_{p,s}: B(y,d(y))\cap B(x,\frac{r}{10})\neq\emptyset\}$.
Hence,

$$\begin{aligned}H^d(E_{p,s}\cap B(x,r)) &\geq H^d(F_{p,s}\cap B(x,\frac{r}{10})) + \theta^{-1}C_0^{-1}H^d((F_p\setminus F_{p,s})\cap B(x,\frac{r}{10}))\\ &\geq \theta^{-1}C_0^{-1}H^d(F_p\cap B(x,\frac{r}{10}))\\ &\geq \theta^{-1}C_0^{-1}\frac{1}{ps}10^{-d}r^d(\text{because } x\in F_{p,s}).\end{aligned}$$

To prove regularity for $x\in C_y$ (with $y\in H_{p,s}$), two facts are useful:

(i) There exists $z\in F_{p,s}$ such that $\operatorname{dist}(x,z)\leq 11d(y)$ (therefore, we can use the preceding case "$x\in F_{p,s}$" for the big scales);

(ii) The piece of d-plane C_y is (locally) d-regular.

□

We conclude this section with a very nice result due to J. M. Marstrand.

THEOREM 13. *Let $E\subset\mathbb{R}^n$ and let $s>0$. Assume that $H^s(E)<+\infty$ and that $\theta^s(x,E)$ exists for H^s-almost every $x\in E$. Then, s must be an integer.*

PROOF. We will give a proof only in the case $E\subset\mathbb{C}$ and $s\in]0,1[$. Namely, we will prove that, if $E\subset\mathbb{C}$ and $s\in]0,1[$ with $H^s(E)<\infty$, then $\theta^s(x,E)$ fails to exist almost everywhere.

Assume that the conclusion is false: there exists a subset of E with $0<H^s(E)<\infty$ such that the density at each point of this subset exists. Note that this density can not be less than $2^{-s}>\frac{1}{2}$ (by proposition 9). Let $\varepsilon>0$. Then, there exists a subset $F\subset E$ (that we can assume to be closed) with $H^s(F)>0$ such that, if $x\in F$, $\theta^s(x,E)$ exists and for any $r\leq\varepsilon$, $H^s(E\cap B(x,r))>\frac{1}{2}(2r)^{-s}$. Let y be an accumulation point of F and let $\eta\in]0,1[$. We denote by $A_{r,\eta}$ the annulus $B(y,(1+\eta)r)\setminus B(y,(1-\eta)r)$. Thus, we have

$$(2r)^{-s}H^s(E\cap A_{r,\eta}) = (2r)^{-s}H^s(E\cap B(y,(1+\eta)r)) - (2r)^{-s}H^s(E\cap B(y,(1-\eta)r)).$$

From this, we get

$$\lim_{r\to 0}\frac{H^s(E\cap A_{r,\eta})}{(2r)^s}=\theta^s(y,E)((1+\eta)^s-(1-\eta)^s).$$

On the other hand, for arbitrarily small r, there exists $x\in F$ such that $|x-y|=r$ (this follows from the fact that y is an accumulation point of F). Therefore, $B(x,\frac{1}{2}r\eta)\subset A_{r,\eta}$. This yields

$$H^s(E\cap A_{r,\eta})\geq H^s(E\cap B(x,\frac{1}{2}r\eta))\geq\frac{1}{2}r^s\eta^s,$$

and then,

$$2^{-(s+1)}\eta^s\leq\theta^s(y,E)((1+\eta)^s-(1-\eta)^s).$$

By taking $\eta\to 0$ in the previous inequality, we get a contradiction. This proof is taken from [**36**], pages 55-56. See [**65**] for a proof in the general case.

□

5. Rectifiable and purely unrectifiable sets

In this section, we restrict to subsets of $\mathbb{C}$. But all the results given are true in higher dimensions.

We say that $E\subset\mathbb{C}$ is 1-rectifiable if there exist Lipschitz maps $f_j:\mathbb{R}\to\mathbb{C}$ such that $H^1(E\setminus\cup_j f_j(\mathbb{R}))=0$. In other words, E can be covered by a countable union of Lipschitz curves (up to a set of zero 1-dimensional Hausdorff measure). For instance, a rectifiable curve is a rectifiable set (be careful about the terminology !). For this, see proposition 21. In fact (see [**36**]), E is 1-rectifiable if and only E can be covered by a countable union of rectifiable Jordan curves (up to set of zero measure).
Remark. By Kirszbraun's theorem (see section 1), a set $E\subset\mathbb{C}$ is rectifiable if and only if there exist subsets E_j, $j\in\mathbb{N}$, of $\mathbb{R}$ and Lipschitz maps $f_j:E_j\to\mathbb{C}$ such that $H^1(E\setminus\cup_j f_j(E_j))=0$.

On the other hand, a set $E\subset\mathbb{C}$ is said to be purely 1-unrectifiable if $H^1(E\cap\Gamma)=0$ for every rectifiable curve $\Gamma\subset\mathbb{C}$. For example, the four corners Cantor set we described in section 2 is purely unrectifiable (see below).

Every set $E\subset\mathbb{C}$ with $H^1(E)<\infty$ can be decomposed in a "good" part and a "bad" part, namely $E=E_{rect}\cup E_{unrect}$ where this union is disjoint, E_{rect} is rectifiable and E_{unrect} is purely unrectifiable. To see this, set $M=\sup H^1(E\cap F)$ where the supremum is taken over all 1-rectifiable sets F in $\mathbb{C}$. Choose for any $j=1,2,..,$ a 1-rectifiable set F_j such that $H^1(E\cap F_j)>M-\dfrac{1}{j}$. Then, it is easy to see that $E_{rect}=\cup_{j=1}^{\infty}F_j$ and $E_{unrect}=E\setminus E_{rect}$ satisfy the properties above. Note that this decomposition is unique up to a set of zero measure (with respect to H^1).

We now describe several characterizations of rectifiable sets in terms of density, existence of tangents and size of projections. A crucial point is that all these characterizations are only valid for sets E with $H^1(E)<\infty$, although this condition is not required in the definition of rectifiability given above.

We start with a very beautiful result due to A. S. Besicovitch (see [**37**] for a proof).

THEOREM 14. *Let $E \subset \mathbb{C}$ such that $H^1(E) < \infty$. Assume that $\theta^1_*(z, E) \geq \frac{3}{4}$ for almost every $z \in E$. Then, E is rectifiable.*

A. S. Besicovitch gave an example of a purely unrectifiable set $E \subset \mathbb{C}$ for which $\theta^1_*(z, E) = \dfrac{1}{2}$. He conjectured that the best upper bound in the previous theorem is $\dfrac{1}{2}$. The best known result is $\dfrac{2+\sqrt{46}}{12}$ and is due to D. Preiss and J. Tiser [**91**].

THEOREM 15. *Let $E \subset \mathbb{C}$ with $H^1(E) < \infty$.*
(i) E is rectifiable if and only if for H^1-almost every $x \in E$, $\theta^1_*(x, E) = \theta^{*1}(x, E) = 1$.
(ii) E is purely unrectifiable if and only if for H^1-almost every $x \in E$, $\theta^1_*(x, E) < 1$.

As a consequence of this theorem, we obtain that the set $E_1 = \{z \in E, \theta^1_*(z, E) = \theta^{*1}(z, E) = 1\}$ (respectively $E_2 = E \setminus E_1$) is rectifiable (respectively purely unrectifiable) if $H^1(E) < \infty$. Thus, this gives an alternative proof of the fact that a set E with $H^1(E) < \infty$ is the disjoint union of a rectifiable set and a purely unrectifiable set.

A set E that satisfies $\theta^1(z, E) = 1$ for H^1-almost every $z \in E$ is called regular in the sense of Besicovitch. Theorem 15 says that rectifiable sets and regular sets in the sense of Besicovitch are the same. This was originally proved by A. S. Besicovitch in the plane, by E. F. Moore for 1-dimensional sets in $\mathbb{R}^n$, by J. M. Marstrand for 2-dimensional sets in $\mathbb{R}^3$. P. Mattila generalized Marstand's proof in higher dimensions. By using the notion of tangent measure, D. Preiss [**90**] proved the more general result that the rectifiability is equivalent to the existence to positive and finite density almost everywhere. See [**65**] for more details and references.

Remark. The notion of Ahlfors-regularity and the notion of regularity in the sense of Besicovitch are quite different. For instance, the four corners Cantor set is Ahlfors-regular, but is not regular in the sense of Besicovitch.

PROOF. We will only prove (i). See [**36**] for a proof of (ii) in $\mathbb{C}$.

Assume that $E \subset \mathbb{C}$ is rectifiable. Recall that this means that E can be covered by a countable union of rectifiable Jordan curves (up to a set of zero measure). Therefore, by Proposition 9, it is enough to prove that a rectifiable Jordan curve Γ is regular in the sense of Besicovitch. To do this, consider $x \in \Gamma$ such that x is not an endpoint of Γ. The point x divides Γ in two subcurves Γ_1 and Γ_2. Then, for r small enough,

$$H^1(\Gamma_i \cap B(x, r)) \geq r \ (i = 1, 2).$$

Hence,

$$H^1(\Gamma \cap B(x, r)) \geq 2r, \text{ thus } \theta^1_*(x, E) \geq 1.$$

Since $\theta^{*1}(x, E) \leq 1$ (by Proposition 9), the proof is complete.

Conversely, let $E \subset \mathbb{C}$ be a set which is regular in the sense of Besicovitch and which satisfies $H^1(E) < +\infty$. Then, $E = E_{rect} \cup E_{unrect}$ where E_{rect} (respectively E_{unrect}) is rectifiable (respectively purely unrectifiable). By Proposition 9, for H^1 almost every $z \in E_{unrect}$, $\theta^1(z, E_{unrect}) = 1$. If $H^1(E_{unrect}) \neq 0$, then Besicovitch's theorem above contradicts the fact that E_{unrect} is purely unrectifiable. Therefore,

$H^1(E_{unrect}) = 0$ and E is rectifiable.

For the convenience of the reader, we also give a more constructive proof (which uses however a very strong result, that is theorem 14). Let $E \subset \mathbb{C}$ be a set which is regular in the sense of Besicovitch. Then, $\theta^1(z, E) = 1$ for H^1-almost every $z \in E$, and therefore, there exists a rectifiable curve Γ such that $H^1(E \cap \Gamma) > 0$. Indeed, if such a curve does not exist, E is purely unrectifiable and by Besicovitch's theorem, $\theta^1_*(z, E) < \frac{3}{4}$ for H^1-almost every $z \in E$, which is impossible.
We can now construct by induction a sequence of rectifiable curves (Γ_i) by the following way:
- $H^1(\Gamma_1 \cap E) \geq \frac{1}{2} \sup\{H^1(\Gamma \cap E); \Gamma \text{ rectifiable curve}\}$;
- Assume that Γ_1,...,Γ_k have been constructed and set $E_k = E \setminus \bigcup_{j=1}^k \Gamma_j$. The curve Γ_{k+1} is a rectifiable curve such that $H^1(\Gamma_{k+1} \cap E_k) \geq \frac{1}{2} \sup\{H^1(\Gamma \cap E_k); \Gamma \text{ rectifiable curve}\}$;
We now have two cases:
- There exists K such that $sup\{H^1(\Gamma \cap E_K); \Gamma \text{ rectifiable curve}\} = 0$. Thus, we stop the construction and we have $H^1(E \setminus \cup_{k=1}^K \Gamma_k) = 0$.
- Such a K does not exist. Since $\sum_k H^1(\Gamma_{k+1} \cap E_k) \leq H^1(E) < \infty$, $\lim_{k\to 0} H^1(\Gamma_{k+1} \cap E_k) = 0$. Assume that $H^1(E \setminus \cup_k \Gamma_k) > 0$. Therefore, there exists a rectifiable curve Γ such that $H^1(\Gamma \cap (E \setminus \cup_k \Gamma_k)) = d > 0$. But, if k is big enough, $H^1(\Gamma_{k+1} \cap E_k) \leq \frac{d}{10}$. Then, for this k, Γ would have been chosen instead of Γ_k in the construction above. Hence, $H^1(E \setminus \cup_k \Gamma_k) = 0$. Note that the Proposition 9 has been used a lot of times. □

For every $z \in \mathbb{C}$, every line L passing through z, $r > 0$ and $\varepsilon > 0$, we define the cone $C(z, r, \varepsilon, L)$ by

$$C(z, r, \varepsilon, L) = \{y \in D(z, r); d(y, L) \leq \varepsilon |y - z|\}.$$

We say that the line L is the approximate tangent line at z to E if $\theta^s_*(z, E) > 0$ and $\lim_{r\to 0} \frac{1}{r} H^1(E \setminus C(z, r, \varepsilon, L)) = 0$ for all $\varepsilon > 0$.

THEOREM 16. *Let $E \subset \mathbb{R}^2$ with $H^1(E) < \infty$.*
(i) E is rectifiable if and only if, for H^1-almost every $x \in E$, there exists an approximate tangent line to E.
(ii) E is purely unrectifiable if and only if for H^1-almost every $x \in E$, an approximate tangent line to E fails to exist.

It is (almost) clear that for any point x of the four corners Cantor set E, you can not find an approximate tangent line. If Γ is a rectifiable curve, the fact that there exists almost everywhere an approximate tangent line to Γ comes from the Rademacher's theorem.

Let G be the set of lines in $\mathbb{C}$ passing through the origin 0. Usually, G is called the Grassmannian of $\mathbb{C}$. We can equip G with a natural measure μ_G, namely the Lebesgue measure on $[0, 2\pi]$. For any $L \in G$, denote by P_L the orthogonal projection on L.

THEOREM 17. *Let $E \subset \mathbb{C}$ with $H^1(E) < \infty$.*
(i) E is rectifiable if and only if for μ_G-almost every line L of G, $H^1(P_L(A)) > 0$ whenever A is a measurable subset of E such that $H^1(A) > 0$.

(ii) E is purely unrectifiable if and only if for μ_G-almost every line $L \in G$, $H^1(P_L(E)) = 0$.

PROOF. To prove the direct sense of (i), it is enough to prove that if $E \subset \Gamma$ where Γ is a rectifiable curve, then $H^1(P_L(E)) > 0$ except perhaps for one line L. To do this, consider $z \in E$ such that $\Theta^1(z, E) = \Theta^1(z, \Gamma) = 1$. Then, for $\varepsilon > 0$, there exists $r > 0$ such that

$$\begin{aligned} H^1(E \cap B(z,r)) &> (1-\varepsilon^2).2r \text{ and} \\ H^1(\Gamma \cap B(z,r)) &< (1+\varepsilon).2r. \end{aligned}$$

From this, we get

$$\begin{aligned} H^1(E \cap B(z,r)) &> (1-\varepsilon)H^1(\Gamma \cap B(z,r)) \text{ then} \\ H^1((\Gamma \setminus E) \cap B(z,r)) &< \varepsilon H^1(\Gamma \cap B(z,r)). \end{aligned}$$

Therefore, we can find $\Gamma_0 \subset \Gamma \cap B(z,r)$ such that

$$H^1(\Gamma_0 \setminus E) < \varepsilon H^1(\Gamma_0) < 2\varepsilon|y-z|,$$

where y and z are endpoints of Γ_0.

Let ϕ so that $\cos\phi > 2\varepsilon$ and let L_ϕ be a line making an angle of ϕ with the line passing through y and z. We denote by P_ϕ the orthogonal projection on L_ϕ. Then, since P_ϕ is 1-Lipschitz,

$$\begin{aligned} H^1(P_\phi(E)) &> |y-z|\cos\phi - H^1(P_\phi(\Gamma_0 \setminus E)) \\ &\geq |y-z|\cos\phi - H^1(\Gamma_0 \setminus E)) \\ &\geq |y-z|(\cos\phi - 2\varepsilon). \end{aligned}$$

Hence, $H^1(P_\phi(E)) > 0$ except for ϕ such that $\cos\phi \leq 2\varepsilon$, for all $\varepsilon > 0$, that is for at most one direction.
Note that our proof implies that if $E \subset \mathbb{C}$ is a set with $H^1(E) < \infty$ such that there exist two lines L_1 and L_2 with $H^1(P_{L_j}(E)) = 0$ $(j = 1, 2)$, then E is purely unrectifiable. By using this result, it is not difficult to see that the four corners Cantor set is purely unrectifiable. See [**36**] for a proof of the remaining directions in (i) and (ii). □

CHAPTER 2

P. Jones' traveling salesman theorem

The classical Traveling Salesman Problem (TSP) consists in computing the length of the shortest path passing through a given finite collection of points in the plane $\mathbb{R}^2$ (these points represent the cities that the traveling salesman should visit). In this chapter, we are interested in a more geometric problem : under which conditions is a compact set $E \subset \mathbb{C}$ contained in a rectifiable curve ? This problem was solved by P. Jones by introducing the "β numbers".
We will also give an overview of G. David and S. Semmes' theory of uniformly rectifiable sets.

1. The β numbers

Let $E \subset \mathbb{R}^n$. For any $x \in \mathbb{R}^n$, any $t > 0$, we define

$$\beta_\infty^E(x,t) = \inf_L \sup_{y \in E \cap B(x,t)} \frac{d(y,L)}{t}$$

where the infimum is taken over all lines L of $\mathbb{R}^n$. By convention, $\beta_\infty^E(x,t) = 0$ if $E \cap B(x,t) = \emptyset$. Most of the time, we will use the notation $\beta_\infty(x,t)$ instead of $\beta_\infty^E(x,t)$. In the particular case $n = 2$, let $S_{x,t}$ be the strip of smallest width which contains $E \cap B(x,t)$. Then, the width of $S_{x,t}$ is equal to $t\beta_\infty(x,t)$. The quantity $\beta_\infty(x,t)$ measures (in a scale invariant way) how well the set E is approximated by lines in $B(x,t)$. Note that if $E \cap B(x,t)$ is contained in a line, then $\beta_\infty(x,t) = 0$. Let $\beta_\infty(E) = \displaystyle\int_{\mathbb{R}^n}\int_0^{+\infty} \beta_\infty(x,t)^2 d\mathcal{L}^n(x)\frac{dt}{t^n}$. In some cases, it is more convenient to use discrete version of $\beta(x,t)$. Let Q be a cube in $\mathbb{R}^n$. Write

$$\beta_\infty^E(Q) = \beta_\infty(Q) = \inf_L \sup_{y \in E \cap 3Q} \frac{d(y,L)}{\mathrm{diam} Q}$$

where the infimum is taken over all lines L of $\mathbb{R}^n$.
Note that there exists $C > 0$ such that

$$C^{-1}\int_{\mathbb{R}^n}\int_0^{+\infty} \beta_\infty(x,t)^2 d\mathcal{L}^n(x)\frac{dt}{t^n} \le \sum_{Q\in\Delta} \beta_\infty(Q)^2 \mathrm{diam} Q \le C\int_{\mathbb{R}^n}\int_0^{+\infty} \beta_\infty(x,t)^2 d\mathcal{L}^n(x)\frac{dt}{t^n}$$

where Δ is the usual family of dyadic cubes in $\mathbb{R}^n$.

Remark. We can also define higher dimensional versions of the β numbers. For this, take the infimum over all d-planes (for some given integer d) in the previous definition.The number d should be seen as the "dimension" of the set E.

Let Γ be a rectifiable curve in $\mathbb{R}^n$. Then, $\Gamma = s([0, l(\Gamma)])$ where s is the parameterization by arc length of Γ (see chapter 1). Recall that s is 1-Lipschitz. By the Rademacher's theorem, s is differentiable almost everywhere on $[0, l(\Gamma)]$ and therefore, Γ admits a tangent (in the sense of calculus) almost everywhere. Thus, we can expect

H. Pajot: LNM 1799, pp. 17–27, 2002.

that $\lim_{t\to 0}\beta^{\Gamma}_{\infty}(s(u),t)=0$ for $\mathcal{L}^1$-almost every $u\in[0,l(\Gamma)]$. In fact, P. Jones proved in [**49**] that $\beta^{\Gamma}_{\infty}(x,t)$ is in L^2 with respect to the measure $d\mathcal{L}^2(x)\frac{dt}{t^n}$. This result is an analog for sets of the theorem of E. M. Stein and A. Zygmund given in the first chapter.

Theorem 18. *Let Γ be a rectifiable curve in R^n. Then,*

$$\beta_{\infty}(\Gamma)=\int_{\mathbb{R}^n}\int_0^{+\infty}\beta_{\infty}(x,t)^2 d\mathcal{L}^n(x)\frac{dt}{t^n}\leq Cl(\Gamma)$$

where $C=C(n)$

Peter Jones' original proof [**50**] works only in $\mathbb{C}$, because it uses complex analysis. We will sketch the proof given in [**80**]. Even if this proof works in higher dimensions, we will consider (for simplicity) only the case $n=2$. Note that theorem 18 (in the special case where Γ is a Lipschitz graph) can also be seen as an easy consequence of Dorronsoro's result given in chapter 1.

Without loss of generality, we can assume that $\Gamma\subset Q_0$ where Q_0 is the unit square. The general case will follow by using translations and dilations. We denote by $s:[0,l(\Gamma)]\to\mathbb{C}$ the parameterization by arc length of Γ.

We now start with some notations and definitions.
We denote by $\Delta(Q_0)$ the dyadic decomposition of Q_0, namely the family of dyadic squares contained in Q_0. Let $Q\in\Delta(Q_0)$. Similarly, we will denote by $\Delta(Q)$ the dyadic decomposition of Q and by $\Delta_k(Q)$ the family of squares of $\Delta(Q)$ of the kth generation.

Let $\{T^{\alpha},\alpha\in\Lambda(Q)\}$ be the set of connected components of $s^{-1}(Q)$.

Write (see figure 2)

$$\Gamma^{\alpha}=s(T^{\alpha});$$

$$S^{\alpha}=[s(x^{\alpha}),s(y^{\alpha})] \text{ where } x^{\alpha} \text{ and } y^{\alpha} \text{ are endpoints of } T^{\alpha};$$

$$b_{\alpha}=\sup_{y\in\Gamma^{\alpha}} d(y,S^{\alpha}),$$

$$b(Q)=\sup_{\alpha\in\Lambda(Q)} b_{\alpha}.$$

Note that $b(Q)$ looks (up to some normalization) like a β number. Set $r(Q)=\inf_L\sup_{y\in E\cap Q} d(y,L)$. Then, $r(3Q)=\beta_{\infty}(Q)\mathrm{diam}Q$.

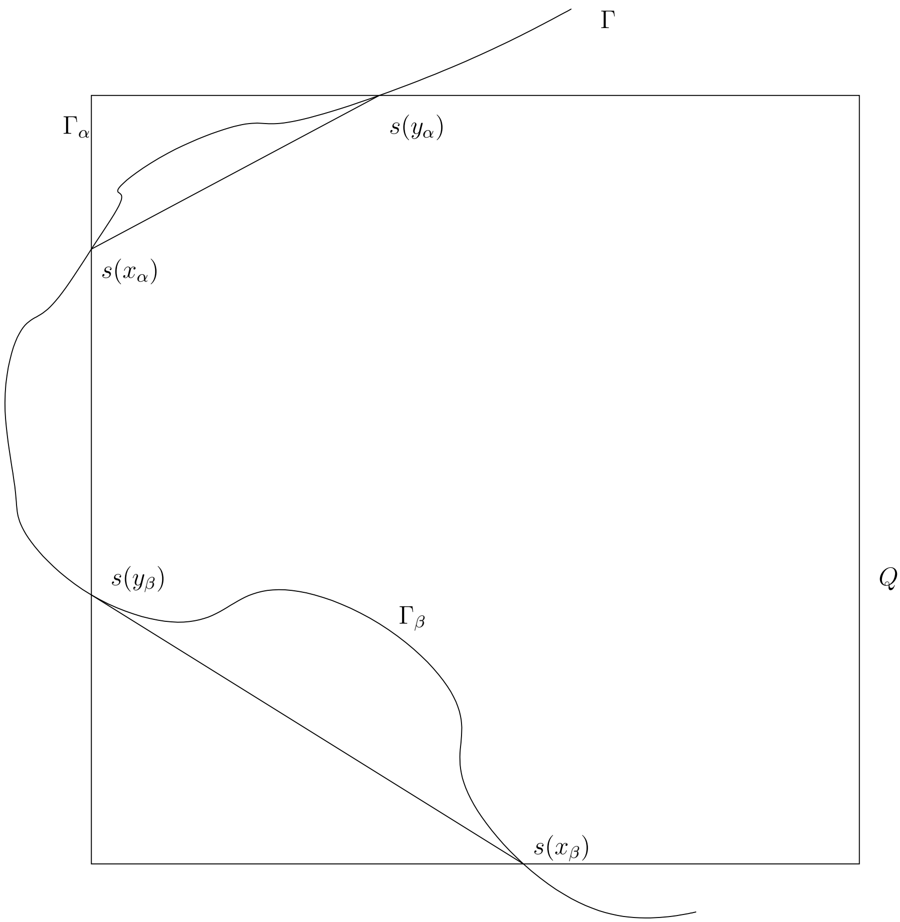

FiGURE 2.

For $\alpha \in \Lambda(Q)$, write

$$\Lambda_{\alpha,k}(Q) = \{\beta; T^\beta \subset T^\alpha \text{ for } \beta \in \Lambda(Q') \text{ and } Q' \in \Delta_k(Q)\},$$

$$t_\alpha = \sup_{x \in S^\beta, \beta \in \Lambda_{\alpha,1}(Q)} d(x, S^\alpha).$$

We claim that $\displaystyle\sum_{R \in \Delta(Q_0), R \subset Q} \frac{b(R)^2}{\text{diam} R} \leq Cl(\Gamma)$ where $C = C(n)$.

Indeed, for any $k = 1, 2, ...$, let $\alpha(k)$ be an index such that $t_{\alpha(k)} = \sup_{\beta \in \Lambda_{\alpha,k}(Q)} t_\beta$. Then, it is clear that, for $\alpha \in \Lambda(Q)$,

$$b_\alpha \leq \sum_{k=0}^{\infty} t_{\alpha(k)}. \tag{9}$$

Moreover, by the Pythagorean theorem, for $\alpha \in \Lambda(Q)$,

$$\frac{t_\alpha^2}{\text{diam } Q} \leq C \left(\left(\sum_{\beta \in \Lambda_{\alpha,1}(Q)} l(S^\beta) \right) - l(S^\alpha) \right). \tag{10}$$

The claim follows from (9), (10) and standard computations.

Consider now

$$\mathcal{A} = \{Q \in \Delta(Q_0); b(Q) \leq \delta r(Q)\},$$
$$\mathcal{B} = \{Q \in \Delta(Q_0); b(Q) > \delta r(Q)\}$$

where $\delta > 0$ should be chosen small enough.
Then,

$$\sum_{Q \in \mathcal{B}} \beta_\infty(Q)^2 \text{diam} Q = \sum_{Q \in \mathcal{B}} \frac{r(3Q)^2}{\text{diam} Q} \leq \frac{1}{\delta^2} \sum_{Q \in \Delta(Q_0)} \frac{b(Q)^2}{\text{diam} Q} \leq C l(\Gamma).$$

The last step of the proof is a little bit technical and quite lengthy. The main idea is the following:
Assume that δ is small enough. Then, if $Q \in \mathcal{A}$, $Q \cap \Gamma$ is almost a finite union of straight segments. Therefore, you can associate to Q a piece of $\Gamma \cap Q$, denoted by E_Q, such that $r(Q) \leq C l(E_Q)$ and $\sum_{Q \in \mathcal{A}} l(E_Q) \leq C l(\Gamma)$. Note that to get the last condition, it is enough to construct the E_Q's such that any point of Γ belongs to E_Q for at most C squares $Q \in \mathcal{A}$. If $\Gamma \cap Q$ is exactly a finite union of segments, such a construction is not hard to do. The general case needs more efforts !
Assuming such a construction has be done, we get

$$\sum_{Q \in \mathcal{A}} \beta_\infty(Q)^2 \text{diam} Q = \sum_{Q \in \mathcal{A}} \frac{r(3Q)^2}{\text{diam} Q} \leq C \sum_{Q \in \mathcal{A}} \frac{l(E_Q)^2}{\text{diam} Q} \leq C \sum_{Q \in \mathcal{A}} l(E_Q) \leq C l(\Gamma).$$

Remark. The right subset of $\Delta(Q_0)$ you have to consider is $\mathcal{A} = \{Q \in \Delta(Q_0); b(Q^*) \leq \delta r(Q)\}$ where Q^* is a well chosen square that contains λQ (The constant $\lambda > 1$ should be carefully chosen). See lemma 1 of [**80**].

Remark. Theorem 18 was originally proved by P. Jones for Lipschitz graphs (see [**49**]). His motivation was to give a geometric proof of the L^2-boundedness of the Cauchy operator on Lipschitz graphs. We will come back later to this problem.

2. Characterization of subsets of rectifiable curves

It turns out that the condition of theorem 18 is also sufficient for a set to be contained in a rectifiable curve.

THEOREM 19. *Let $E \subset \mathbb{R}^n$ be a compact set. Assume that $\beta_\infty(E) < \infty$. Then, there exists a rectifiable curve $\Gamma \subset \mathbb{R}^n$ such that*

(i) $E \subset \Gamma$;
(ii) $l(\Gamma) \leq C(\beta_\infty(E) + \textit{diam } E)$.

Theorem 18 and theorem 19 together give a characterization of subsets of rectifiable curves in $\mathbb{R}^n$ and, by modifying a little the proofs, of subsets of regular curves.

THEOREM 20. *Let $E \subset \mathbb{R}^n$ be a compact set.*
(i) The set E is contained in a rectifiable curve Γ if and only if $\beta_\infty(E) < \infty$.
(ii) The set E is contained in a Ahlfors regular curve Γ if and only if for any $x \in E$, any $R \in (0, diamE)$,

$$\int_{E\cap B(x,R)} \int_0^R \beta_\infty(y,t)^2 dH^1(y)\frac{dt}{t} \leq CR.$$

Note that the quadratic condition in (ii) is a Carleson type condition. More precisely, it means that $\beta_\infty(x,t)^2 dH^1(x)\frac{dt}{t}$ is a Carleson measure on $E \times \mathbb{R}^+$. Since E is Ahlfors-regular, the discussion of the first section of chapter 1 makes sense in this setting.

We now sketch the construction of Γ given by Peter Jones in [**50**]. For this, we start with a well known result (see [**29**] for a proof).

PROPOSITION 21. *Let $A \subset \mathbb{R}^n$ be a continuum (that is a compact connected subset of $\mathbb{R}^n$). Assume that $H^1(A) < \infty$. Then, there exists a Lipschitz curve Γ such that $A \subset \Gamma$ and $H^1(A) \leq l(\Gamma) \leq CH^1(A)$ (where $C > 1$ is a constant that does not depend on A).*

Hence, to prove theorem 19, it is enough to construct a continuum Γ that satisfies (i) and (ii). Note that proposition 21 has no counterpart in higher dimensions. For instance, a compact connected surface in $\mathbb{R}^n$ is not in general contained in a Lipschitz surface. This explains that it is hard to give an analog of theorem 19 in higher dimensions. Indeed, you have to construct "by hands" a nice parameterization of the surface Γ that contains E (see [**85**]), whereas the Lipschitz parameterization of the curve Γ is given for free by proposition 21.
Without loss of generality, we can assume that diam$E = 1$. For any $k \in \mathbb{N}$, consider a set $\Delta_k \subset E$ such that
- If x, y are in Δ_k, then $|x - y| \geq 2^{-k}$;
- For any $y \in E$, there exists $x \in \Delta_k$ such that $|x - y| \leq 2^{-k}$.
Choose x_0 and y_0 in E such that $|x_0 - y_0| =$ diam$E$ and let Γ_0 be the segment $[x_0 - 2y_0, y_0 - 2x_0]$. Assume by induction that polygonal lines Γ_0, ..., Γ_k have been constructed with $\Delta_j \subset \Gamma_j$ and with some other additional properties (that we do not need for our description of P. Jones' construction). Note that all the endpoints of segments of Γ_0, ..., Γ_k are not in general in E. The continuum Γ will be the limit (in a reasonable sense) of the sequence (Γ_k). Let $x \in \Delta_{k+1} \setminus \Delta_k$. We would like to construct a new polygonal line (denoted by Γ^1_{k+1}) passing through x by "deforming" Γ_k. The point is to control $l(\Gamma^1_{k+1}) - l(\Gamma_k)$ by the β-numbers.
Assume first that $\beta_\infty(x, C2^{-k}) \geq \varepsilon_0$ (where ε_0 is small enough, see below). Then, Γ^1_{k+1} is obtained by connecting x to all points of $\Delta_k \cap B(x, C2^{-k})$. Then,

$$\begin{aligned} l(\Gamma^1_{k+1}) - l(\Gamma_k) &\leq C2^{-k} \\ &\leq C\varepsilon_0^{-2}\beta^2_\infty(x, C2^{-k})2^{-k}. \end{aligned}$$

From now on, we assume that $\beta_\infty(x, C2^{-k}) \leq \varepsilon_0$. Roughly speaking, this means that $E \cap B(x, C2^{-k})$ is almost contained in a line (if ε_0 is small enough). Let $[y, z]$ be the straight segment of Γ_k which is the closest to x. The most favorable case is the following:
(a) y, z are in E;

(b) The orthogonal projection of x on the line passing through y and z is in the segment $[y,z]$.
Assume now that (a) and (b) hold.
Case A: The point x is almost in the middle of y and z, that is $K^{-1}|x-z| \leq |x-y| \leq K|x-z|$ for some well chosen constant $K>0$. The new curve Γ_{k+1}^1 is obtained by replacing $[y,z]$ by $[y,x]\cup[x,z]$. Note that the distance from x to the line passing through y and z is comparable to $\beta(x, C2^{-k})2^{-k}$. Thus, by the Pythagorean theorem, we get

$$l(\Gamma_{k+1}^1) - l(\Gamma_k) \leq C\beta_\infty^2(x, C2^{-k})2^{-k}.$$

Case B: The point x is closed to one endpoint of $[y,z]$ (for instance y), that is $|x-z| \geq K|x-y|$. The new curve Γ_{k+1}^1 is obtained by adding the segment $[y, 2x-y]$. Therefore, if K is big enough,

$$l(\Gamma_{k+1}^1) - l(\Gamma_k) << \frac{1}{3} l([y,z]).$$

In fact, we prove that the sum of the lengths of all the segments (with endpoints y or z) constructed by using case B is less that $\frac{1}{3}|y-z|$ (if K is big enough). Note that $[y,z]$ will be contained in the final curve Γ.

The other cases can be treated in the same manner. Applying this strategy to the other points of $\Delta_{k+1}\setminus\Delta_k$, we get a new curve Γ_{k+1} which contains Δ_{k+1}. Then, by taking the limit of the sequence (Γ_j) (with respect to the Hausdorff topology), we get a curve Γ which contains E. Moreover,

$$\begin{aligned} l(\Gamma) &\leq \sum_{j\in\mathbb{N}} (l(\Gamma_{j+1}) - l(\Gamma_j)) + l(\Gamma_0) \\ &\leq C\sum_{j\in\mathbb{N}}\sum_{x\in\Delta_j} \beta_\infty(x, C2^{-j}) + \frac{1}{3} l(\Gamma) + C\mathrm{diam}E. \end{aligned}$$

The first term comes from case A, the second from case B and the third is $l(\Gamma_0)$. From this, we get

$$l(\Gamma) \leq C(\beta_\infty(E) + \mathrm{diam}E).$$

In [**7**], C. Bishop and P. Jones established a "local" version of theorem 19.

THEOREM 22. *Let E be a compact set in $\mathbb{R}^n$. Assume that there exists $M>0$ such that $\int_0^{diam E} \beta_\infty(x,t)^2 \frac{dt}{t} \leq M$ for every $x\in E$. Then, there exists a rectifiable curve Γ such that:*

(i) $E\subset\Gamma$,

(ii) $l(\Gamma) \leq Ce^{CM}$ *diam* E.

Remark. A slight modification of the proof of theorem 22 shows that a compact set $E\subset\mathbb{R}^n$ satisfying the hypothesis of theorem 22 (that is there exists $M>0$ such that $\int_0^{\mathrm{diam}E} \beta_\infty(x,t)^2 \frac{dt}{t} \leq M$ for every $x\in E$) is contained in an Ahlfors-regular curve whose regularity constant depends on M.

The construction of Γ is the same as in the proof of theorem 19. As previously, the crucial point is to control the length $l(\Gamma)$ in terms of β. To do this, we construct functions f_k supported on Γ_k (with the same notation as above) such that

(i) $0 \le f_k(x) \le C + C\sum_Q \beta_\infty(Q)^2$ where the sum is over all the dyadic squares $Q \in \Delta$ that contain x.

(ii) $\displaystyle\int_{\Gamma_k} e^{-f_k} ds \le C.$

From this, we easily deduce that $l(\Gamma) \le Ce^{CM}\text{diam}E$. We now explain the construction of f_k. Let $f_0 = 0$ and assume that f_k is given. Recall that there are (almost) two cases in the construction of Γ_{k+1} from Γ_k. With the same notations as above, if we assume that x is closer to y, there exists a dyadic square Q of side length 2^{-k+1} such that x, $y \in 3Q$. This follows from the fact that $x \in \Delta_{k+1} \setminus \Delta_k$.

Case A: $K^{-1}|x-z| \le |x-y| \le K|x-z|$.
Define $f_{k+1}(\xi) = f_k(\xi) + C\beta_\infty(Q)^2\chi_{3Q}(\xi)$. Hence, since $|y-x|+|x-z|-|y-z| \le C\beta_\infty(Q)^2\text{diam}Q$, we get

$$\int_{\Gamma_{k+1}} e^{-f_{k+1}} ds \le \int_{\Gamma_k} e^{-f_k} ds \le C.$$

Case B: $|x-z| \ge K|x-y|$.
Let I be the middle third interval of $[y,z]$. Define

$$f_{k+1}(\xi) = f_k(\xi) + C\beta_\infty(Q)^2\chi_{3Q}(\xi) + C\chi_I(\xi).$$

Since $l(I) >> \text{diam}Q$ if K is big enough, we get

$$\int_{\Gamma_{k+1}} e^{-f_{k+1}} ds \le \int_{\Gamma_k} e^{-f_k} ds \le C.$$

A version of theorem 22 in higher dimensions will be given in the next section.

Theorem 22 is a partial result toward the so-called Carleson ε^2 conjecture: Let Γ be a closed Jordan curve and let Ω_i, $i = 1,2$, the connected components of $\mathbb{C} \setminus \Gamma$. For every $x \in \Gamma$, every $t > 0$, denote by $\Theta_i(x,t)$ the angular measure of the largest component of $\Omega_i \cap \partial B(x,t)$ and write $\varepsilon(x,t) = \max_{i=1,2} |\pi - \Theta_i(x,t)|$. Thus, if Γ is a line, $\theta_i(x,t) = \pi$ and $\varepsilon(x,t) = 0$. Note that $\varepsilon(x,t)$ can be small whereas $\beta_\infty(x,t)$ is big. L. Carleson conjectures that Γ is rectifiable if and only if $\displaystyle\int_0^{\text{diam}\Gamma} \varepsilon^2(x,t)\frac{dt}{t} < +\infty$ for almost every $x \in \Gamma$. Only the direct sense has been proved (see [**7**] for a proof using the harmonic measure).

We conclude this section with a very useful corollary of theorems 19 and 21 (see [**9**] for a proof and [**8**], [**10**], [**48**] for applications).

THEOREM 23. *Let $E \subset \mathbb{R}^n$ be a continuum. Assume that $\beta_\infty(x,t) \ge \beta_0$ for every $x \in E$, every $t \in]0, diamE[$. Then, $dim_H E \ge 1 + C\beta_0^2$ (for some absolute constant $C > 0$).*

3. Uniformly rectifiable sets

The notion of uniformly rectifiable set introduced by G. David and S. Semmes can be seen as a quantitative version of the notion of rectifiable set.

Let E be a Ahlfors regular set with dimension 1 in $\mathbb{C}$. Then, E can be equipped with a family of "dyadic cubes" (see [**25**]) which plays the same role than the family of usual dyadic intervals in $\mathbb{R}$.

PROPOSITION 24. *There exists a family $\Delta(E)$ of partitions $\Delta_j(E)$, $j \in \mathbb{Z}$, of E by "dyadic cubes" $Q \subset E$ such that*

(i) *If $j \geq k$, $Q \in \Delta_j(E)$, $Q' \in \Delta_k(E)$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$;*

(ii) *If $Q \subset \Delta_j(E)$ then*

$$C^{-1}2^{-j} \leq diam\ Q \leq C2^{-j},$$

$$C^{-1}2^{-j} \leq H^1(Q) \leq C2^{-j}.$$

(iii) *If $Q \in \Delta(E)$, then there exists $c_Q \in E$ such that $B(c_Q, C^{-1}diamQ) \cap E \subset Q$.*

Moreover, if $Q \in \Delta_j(E)$ for some j, the number of "sons" of Q (that is the number of cubes $Q' \in \Delta_{j+1}(E)$ with $Q' \subset Q$) is bounded (independently of j).

Remarks. Cubes can be also build to have a "small" boundary, that is, for any $Q \in \Delta(E)$, for any $0 < \tau < 1$,

$$H^1(\{z \in Q; d(z, E \setminus Q) \leq \tau \mathrm{diam} Q\}) \leq C\tau^{\frac{1}{C}} H^1(Q).$$

This property is very useful to get estimates for singular integral operators (see chapter 6). The proposition above has been extended to space of homogeneous type by M. Christ in [**18**] (where a very elegant construction is given).

We now define L^q version of the β_∞'s.
If $q \geq 1$, $x \in \mathbb{C}$ and $t > 0$, write

$$\beta_q(x,t) = \inf_L \left(\frac{1}{t} \int_{y \in E \cap B(x,t)} \left(\frac{d(y,L)}{t} \right)^q dH^1(y) \right)^{\frac{1}{q}},$$

and if $Q \in \Delta(E)$,

$$\beta_q(Q) = \inf_L \left(\frac{1}{\mathrm{diam} Q} \int_Q \left(\frac{d(y,L)}{t} \right)^q dH^1(y) \right)^{\frac{1}{q}}$$

where the infimum is taken over all lines L in $\mathbb{C}$.

Before stating a result of G. David and S. Semmes, we need a definition. We say that E admits a corona decomposition if, for any $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that $\Delta(E)$ can be divided into a bad part $\mathcal{B}$ and a good part $\mathcal{G}$ which satisfy:
(i) The bad set $\mathcal{B}$ is not too big, in the sense that it satisfies a Carleson packing condition, that is, for $R \in \Delta(E)$,

$$\sum_{Q \in \mathcal{B}, Q \subset R} \mathrm{diam} Q \leq C \mathrm{diam} R,$$

(ii) The good set $\mathcal{G}$ can be partitioned into a family $\mathcal{F}$ of subsets S of $\mathcal{G}$ such that:

- For any $S \in \mathcal{F}$, there exists a top cube $Q(S)$ such that, if $Q \in S$ and if $Q' \in \Delta(E)$ with $Q \subset Q' \subset Q(S)$ then, $Q' \in S$ (property of "coherence" of S);

- If $S \in \mathcal{F}$, E is well approximated by a Lipschitz graph Γ on S, that is, for any $Q \in S$, any $x \in E$ with $d(x, Q) \leq \mathrm{diam} Q$, $d(x, \Gamma) \leq C\varepsilon \mathrm{diam} Q$;

- The regions (called stopping time regions) S are not too many: for any $R \in \Delta(E)$,

(11)
$$\sum_{S\in\mathcal{F}, Q(S)\subset R} \mathrm{diam} Q(S) \le C \mathrm{diam} R.$$

(or equivalently $\sum_{S\in\mathcal{F}, Q(S)\subset R} H^1(Q(S)) \le CH^1(R)$.).

Fix $q \in [1, +\infty]$.

THEOREM 25. [**28**] *Let $E \subset \mathbb{C}$ be an Ahlfors regular set of dimension 1. Then, the following conditions are equivalent.*
(i) E is contained in an Ahlfors-regular curve Γ;
(ii) There exists $C > 0$ such that, for all $x \in E$, all $R \in]0, diamE[$,

$$\int_{B(x,R)\cap E} \int_0^R \beta_q(x,t)^2 dH^1(x) \frac{dt}{t} \le CR.$$

(iii) E admits a corona decomposition.
(iv) There exists a bilipschitz map $f : \mathbb{R} \to \mathbb{C}$ such that $E \subset f(\mathbb{R})$;

A set that satisfies one of the condition (i)-(iv) is said to be uniformly rectifiable. A typical uniformly rectifiable set is a Lipschitz graph. Note that the previous theorem has an analogue in higher dimensions and there are a lot of other characterizations of uniformly rectifiable sets (see [**28**], [**29**]).

By Tchebychev, the condition (ii) with $q = \infty$ implies that E satisfies the weak geometric lemma, that is for each $\varepsilon > 0$ the set $A_\varepsilon = \{(x,t) \in E \times \mathbb{R}^+; \beta_\infty(x,t) > \varepsilon\}$ is a Carleson set. By the discussion of the first chapter, this means that the set A_ε is "small".

Peter Jones's construction can be used to prove $(ii) \Leftarrow (i)$ in the case $q = \infty$ (see theorem 20). But, in the general case, his construction should be adapted and it is not so easy (see [**86**]).

We now explain why a uniformly rectifiable set is rectifiable (in the sense of the first chapter). For this, consider an Ahlfors-regular set $E \subset \mathbb{C}$ which satisfies conditions (i)-(iv). Fix $R \in \Delta(E)$. With the same notation as above, denote by $h(y)$ the number of stopping time regions $S \in \mathcal{F}$ so that $y \in S$ and $S \subset R$. Recall that the Carleson packing condition $\sum_{S\in\mathcal{F}, Q(S)\subset R} H^1(Q(S)) \le CH^1(R)$ holds. Let $M > 0$ and set $F_M(R) = \{y \in R; h(y) \ge M\}$. Then, $\sum_{S\subset\mathcal{F};Q(S)\subset R} H^1(Q(S)) \ge MH^1(F_M(R))$. Thus, for any $M > 0$, $H^1(F_M(R)) \le \frac{C}{M} H^1(R)$. This implies that almost every $y \in R$ belongs to a finite number of stopping time regions. Therefore, H^1-almost every $x \in E$ belongs to a finite number of stopping time regions. Hence, for H^1-almost $x \in E$, there exist $S \in \mathcal{F}$, $t_S(x) > 0$ such that $(x,t) \in S$ for any $t \in [0, t_S(x)[$, so $x \in \Gamma_S$. Since the set of stopping regions S such that $H^1(\{x \in E; (x,0) \in S\}) > 0$ is countable, E can be covered by a countable union of Lipschitz graphs (with uniformly bounded Lipschitz constants).

The proof of theorem 25 is quite long. We will only sketch the proof of $(ii) \Rightarrow (iii)$ in the case $q = \infty$. To do this, choose $\varepsilon_0 > 0$ small enough. We denote by $\beta(Q) = \beta_\infty(kQ)$ (where $k > 1$ is well chosen). Write

$$\mathcal{B} = \{Q \in \Delta(E); \beta(Q) \geq \varepsilon\}.$$

Let $Q_0 \in \Delta(E) \setminus \mathcal{B}$ and let S_0 be the set of "dyadic cubes" $Q \in \Delta(E)$ such that

- $Q \subset Q_0$;
- For any $\tilde{Q} \in \Delta(E)$ such that $Q \subset \tilde{Q} \subset Q_0$, $\tilde{Q} \in \Delta(E) \setminus \mathcal{B}$;
- Any brother of Q belongs to $\Delta(E) \setminus \mathcal{B}$.

For any cube $Q \in S_0$, there exists a line L_Q minimizing $\beta(Q)$. Let S be the set of "dyadic cubes" $Q \in S_0$ such that, for any $\tilde{Q} \in S_0$ with $Q \subset \tilde{Q} \subset Q_0$ or any brother $\tilde{Q}$ of Q, angle$(L_{Q_0}, L_{\tilde{Q}}) \leq \eta$. The region S is the stopping time region with top cube Q_0.
Using this procedure, it is not difficult to construct the required stopping time regions. We now explain how to construct a Lipschitz graph $\Gamma(S)$ associated to the stopping time region S. Let $Q(S)$ be the top cube of S. Denote by Π the orthogonal projection on $L_{Q(S)}$ and by $\Pi^\perp$ the orthogonal projection on $L^\perp_{Q(S)}$. The coordinate system is now $x = (\Pi(x), \Pi^\perp(x))$. For any $Q \in S$, the line L_Q is the graph of an affine function a_Q. Denote by $m(S)$ the set of minimal cubes of S. Using a partition of unity, we can construct a function f_S by gluing together the affine functions a_Q, $Q \in m(S)$.
For $x \in E$, write $d(x) = \inf_{Q\in S}(\mathrm{diam}Q + d(x, Q))$. If $d(x) = 0$, then $x = (\Pi(x), \Pi^\perp(x))$. If x and y are two points such that $d(x) = 0$ and $d(y) = 0$, then

$$|\Pi^\perp(x) - \Pi^\perp(y)| \leq 2\eta|\Pi(x) - \Pi(y)| \leq 2\eta|x - y|. \tag{12}$$

This yields $|f_S(x) - f_S(y)| \leq C\eta|x - y|$. The first estimate of (12) follows from the existence of a square Q with size comparable to $|x - y|$, closed to x and y, and such that angle$(L_Q, L_{Q(S)}) \leq \eta$.
More generally, if x and y are two points of S such that $|x - y| \geq \inf(d(x), d(y))$, the previous estimate (12) still holds for the same reason.
Therefore, f_S is Lipschitz and its graph $\Gamma(S)$ satisfies $d(x, \Gamma(S)) \leq C\varepsilon_0 d(x)$ whenever $x \in S$. The last point follows easily from the construction of the stopping time region S.

The main point to prove the Carleson estimate (11) is to show that, for each stopping time regions S, one of these three possibilities occurs:

(P1) $H^1(\{x \in Q(S); d(x) = 0\}) \geq C^{-1}H^1(Q(S))$;

(P2) $\sum_{Q\in m^*(S)} H^1(Q) \geq C^{-1}H^1(Q(S))$ where $m^*(S)$ is the set of minimal cubes in S which have one child in $\mathcal{B}$;

(P3) $\sum_{Q\in S} \beta_\infty(Q)^2 \geq \alpha H^1(Q(S))$ where $\alpha > 0$ is small enough.

Roughly speaking, regions which satisfy (P1) (respectively (P2)) (respectively (P3)) are regions S where mostly you never stop (respectively you stop because $\beta_\infty((Q) \geq \varepsilon)$ (respectively you stop because angle$(L_{Q(S)}, L_Q) \geq \eta$).
Here, the delicate point consists in showing that a stopping time region which does not satisfy (P1) and (P2) satisfies (P3). For this, we need precise estimates of $|\nabla f_S|$ and we use J. Dorronsoro's theorem given in chapter 1.

The stopping time regions for which (P1) holds satisfy the Carleson estimate (11) because the sets of points $\{x \in Q(S), d(x) = 0\}$ are disjoint. The Carleson condition (11) for stopping time regions which satisfy (P2) follows from the Carleson estimate for the betas. The Carleson estimate for the stopping regions with (P3) follows from the following estimate (which is a consequence of the Carleson estimate for the betas):

$$\sum_{Q \subset B(x,R)} \beta_\infty(Q)^2 \leq \int_{t \in]0,R[} \int_{E \cap B(x,R)} \beta_\infty(y,t)^2 dH^1(y) \frac{dt}{t} \leq CR.$$

Using theorem 25 and the covering theorem by Ahlfors regular sets given in chapter 1, quantitative conditions of rectifiability can be given. This result has an analog in higher dimensions (that is for set $E \subset \mathbb{R}^n$ with $H^d(E) < +\infty$ for some $d \in \mathbb{N}$).

THEOREM 26. [**87**] *Let $E \subset \mathbb{C}$ be a compact set and let $q \in [1, +\infty]$.*
(i) If, for H^1-almost all $x \in E$, $\Theta^1_*(x, E) > 0$ and $\int_0^1 \beta_q(x,t)^2 \frac{dt}{t} < \infty$, then E is 1-rectifiable.
(ii) If E is Ahlfors-regular with dimension 1, E is 1-rectifiable if and only if, for H^1-almost every $x \in E$, $\int_0^1 \beta_q(x,t)^2 \frac{dt}{t} < \infty$.

CHAPTER 3

Menger curvature

In [**71**], M. Melnikov rediscovered the Menger curvature and underlined its relationship with the Cauchy integral (see (13) below). In this chapter, we will see that the Menger curvature is also a useful tool to study geometric properties of sets and measures in $\mathbb{C}$. The last sections of this chapter will be devoted to the (hard) problem of giving estimates of the Menger curvature of some measures.

1. Definition and basic properties

Let x, y and z be three non collinear points in $\mathbb{C}$ (in particular, x, y and z are distinct). Then, the Menger curvature $c(x,y,z)$ of x, y and z is the inverse of the radius of the circle passing through x, y and z. If x, y and z are collinear, we set $c(x,y,z) = 0$.

PROPOSITION 27. *Let z_1, z_2 and z_3 be three points in $\mathbb{C}$. Then,*

$$(13)\quad c(z_1,z_2,z_3)^2 = \left(\frac{4S(z_1,z_2,z_3)}{|z_1-z_2||z_2-z_3||z_3-z_1|}\right)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)}-z_{\sigma(2)})\overline{(z_{\sigma(1)}-z_{\sigma(3)})}},$$

where σ runs through the set of permutations of $\{1,2,3\}$ and $S(z_1,z_2,z_3)$ is the area of the triangle with vertices z_1, z_2 and z_3.

This formula is magic, since it shows that the sum over σ is real and nonnegative ! We leave the proof of proposition 27 to the reader as an exercise of elementary geometry.

From proposition 27 (or from its proof), we get two other ways to compute $c(x,y,z)$:

$$(14)\qquad c(x,y,z) = \frac{2d(x,L_{yz})}{|x-y||x-z|}$$

where L_{yz} is the line through y and z.

$$(15)\qquad c(x,y,z) = 2\frac{\sin\alpha}{L}$$

where L is the length of a side of the triangle with vertices x, y, z and α is the angle opposite to that side.

Let μ be a positive Borel measure in $\mathbb{C}$. We define its Menger curvature as the quantity

$$c^2(\mu) = \int\int\int c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z).$$

If $E \subset \mathbb{C}$ with $H^1(E) < +\infty$, we set $c^2(E) = c^2(H^1_{\lceil E})$ where $H^1_{\lceil E}$ denotes the restriction of H^1 to E.

H. Pajot: LNM 1799, pp. 29–54, 2002.

Let $A=\{(x,y,z)\in\mathbb{C}^3; |x-y|\leq|x-z|\leq|y-z|\}$. It is very useful to note that

$$c^2(\mu)\leq 6\int\int\int_A c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z). \tag{16}$$

Observe that, if x, y and z are three points in $\mathbb{C}$ such that $|x-y|\leq|x-z|\leq|y-z|$ and $|x-y|$ is comparable to $|x-z|$, then $c(x,y,z)\leq C\frac{\beta_\infty(x,|y-z|)}{|y-z|}$ where $C>0$ depends on the previous constants. This remark suggests that the Menger curvature can be used to study flatness properties of sets in $\mathbb{C}$.

Remark. The triangle with vertices x, y and z may be "thin", whereas $c(x,y,z)$ is not "small". For instance, for any $\varepsilon>0$, $c(-1,1,e^{i\varepsilon})=1$! In other words, in the previous observation, the fact that the distances $|x-y|$ and $|x-z|$ are comparable is crucial. We will go back later to the comparison between the Menger curvature and the betas.

THEOREM 28. *Let $E\subset\mathbb{C}$ with $H^1(E)<+\infty$. If $c^2(E)<+\infty$, then E is rectifiable.*

This result was originally proved by G. David (unpublished) by adapting P. Jones's construction given in the previous chapter. In [**56**], J. C. Léger proved the theorem by using David-Semmes' corona construction. The difficulties come from the fact that we do not assume that E satisfies some density properties. J. C. Léger's proof is based on the following result.

PROPOSITION 29. *For every $C_0>0$, there exists $\eta>0$ such that, if μ is a compactly supported measure on $\mathbb{C}$ that satisfies*

(i) $\mu(B(0,2))\geq 1$ *and* $\mu(\mathbb{C}\setminus B(0,2))=0$*;*
(ii) $\mu(B)\leq C_0$ *diam B whenever B is a ball in $\mathbb{C}$;*
(iii) $c^2(\mu)\leq\eta$*,*

then there exists a Lipschitz graph Γ such that $\mu(\Gamma)\geq\frac{99}{100}\mu(\mathbb{C})$*.*

The theorem 28 follows easily from this proposition. Indeed, since $H^1(E)<+\infty$, we can write $E=E_{rect}\cup E_{unrect}$ where E_{rect} and E_{unrect} are disjoint and respectively rectifiable and purely unrectifiable. If $H^1(E_{unrect})\neq 0$, then we can find a piece $F\subset E_{unrect}$ which satisfies (up to some rescaling) the hypothesis of the proposition 29 (and with $H^1(F)>0$). This follows from a standard uniformization procedure (see [**36**]). Therefore there exists a rectifiable curve Γ such that $H^1(E_{unrect}\cap\Gamma)>0$. This contradicts the fact that E_{unrect} is purely unrectifiable. Thus, $H^1(E_{unrect})=0$ and E is rectifiable.

Note that proposition 29 is not true if we do not assume that η depends on C_0. For instance, let μ be the restriction of $\mathcal{L}^2$ on the unit disc $D(0,1)$. Then, μ satisfies (i), (ii) and $c^2(\mu)<+\infty$, but for any Lipschitz graph Γ, $\mu(\Gamma)=0$.

Remark. Recently, Y. Lin and P. Mattila [**59**] proved that, if $E\subset\mathbb{C}$ is Ahlfors s-regular (with $0<s\leq\frac{1}{2}$) and satisfies

$$\int_E\int_E\int_E c(x,y,z)^{2s}dH^s(x)dH^s(y)dH^s(z)<+\infty,$$

then there exists a countable family of C^1 curves Γ_1, Γ_2, ... such that

$$H^s\left(E \setminus \left(\bigcup_{j=1}^{+\infty} \Gamma_j\right)\right) = 0.$$

This result is false if $\frac{1}{2} < s < 1$.

2. Menger curvature and Lipschitz graphs

In this section, we give an estimate due to M. Melnikov and J. Verdera [**72**] of the Menger curvature of a Lipschitz graph. This result will allow us to give in the next chapter a geometric proof of the L^2 boundedness of the Cauchy operator on Lipschitz graphs.

THEOREM 30. *Let Γ be the graph of a K-Lipschitz function $f : I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$. Then, $c^2(\Gamma) \leq C(1+K)^{\frac{3}{2}} K^2 diam I$.*

The proof is based on a localization of f and on the following consequence of Plancherel's theorem:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left| \frac{\frac{a(y)-a(x)}{y-x} - \frac{a(z)-a(y)}{z-y}}{z-x} \right|^2 dxdydz \leq C \int_{\mathbb{R}} |a'(t)|^2 dt$$

whenever a is a locally integrable function with $a' \in L^2(\mathbb{R})$.
Remark. Throughout all the proof, we will use the notation dx instead of $d\mathcal{L}^1(x)$.

PROOF. Recall that, if x, y and z are points of $\mathbb{C}$,

$$c(x,y,z) = \frac{2d(x, L_{y,z})}{|x-y||x-z|}$$

where $L_{y,z}$ is the line of $\mathbb{C}$ passing through y and z.
If r,s and t are in I, we get, for $x = (r, f(r))$, $y = (s, f(s))$, and $z = (t, f(t))$,

$$\begin{aligned} c^2(x,y,z) &\leq \left(\frac{(f(t)-f(r))(s-r) - (f(s)-f(r))(t-r))}{|x-y||x-z||y-z|}\right)^2 \\ &\leq \left(\frac{\frac{f(t)-f(r)}{t-r} - \frac{f(s)-f(r)}{s-r}}{t-s}\right)^2 \end{aligned}$$

Write $I = [a,b]$ and set $g(t) = f(t) - \left(f(a) + (t-a)\frac{f(b)-f(a)}{b-a}\right)$ if $t \in I$ and $g(t) = 0$ otherwise. Then, g is a $2K$-Lipschitz function whose derivative is in $L^2(\mathbb{R})$. Moreover, if r, s and t are in I,

$$c^2(x,y,z) \leq \left(\frac{\frac{g(t)-g(r)}{t-r} - \frac{g(s)-g(r)}{s-r}}{t-s}\right)^2.$$

This yields

$$\begin{aligned} c^2(\Gamma) &\leq \int_I\int_I\int_I \left(\frac{\frac{f(t)-f(r)}{t-r} - \frac{f(s)-f(r)}{s-r}}{t-s}\right)^2 (1+|f'(r)|)^{\frac{1}{2}}(1+|f'(s)|)^{\frac{1}{2}}(1+|f'(t)|)^{\frac{1}{2}} drdsdt \\ &\leq (1+K)^{\frac{3}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{f(t)-f(r)}{t-r} - \frac{f(s)-f(r)}{s_r}\right)^2 dr\frac{dsdt}{(s-t)^2} \\ &\leq (1+K)^{\frac{3}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{f(r+h)-f(r)}{h} - \frac{f(r+k)-f(r)}{k}\right)^2 dr\frac{dhdk}{(h-k)^2}. \end{aligned}$$

By using Plancherel's theorem, we get

$$c^2(\Gamma) \leq (1+K)^{\frac{3}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\frac{e^{-i\xi h}-1}{h} - \frac{e^{-i\xi k}-1}{k}\right|^2 |\hat{g}(\xi)|^2 d\xi \frac{dhdk}{(h-k)^2}.$$

Thus, by using the changes of variables $h \to \xi h$ and $k \to \xi k$, and again Plancherel's theorem,

$$\begin{aligned} c^2(\Gamma) &\leq (1+M)^{\frac{3}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\frac{e^{-ih}-1}{h} - \frac{e^{-ik}-1}{k}\right|^2 \frac{dhdk}{(h-k)^2} \int_{\mathbb{R}} |\xi|^2 |\hat{g}(\xi)|^2 d\xi \\ &\leq C(1+K)^{\frac{3}{2}} \int_{\mathbb{R}} |g'(t)|^2 dt \\ &\leq C(1+K)^{\frac{3}{2}} K^2 \mathrm{diam} I. \end{aligned}$$

□

3. Menger curvature and β numbers

In this section, we will compare β numbers and the Menger curvature. All these results have been announced by P. Jones in lectures given at the Universitat Autonoma de Barcelona, but have never been published. All the proofs we will give follow P. Jones' ideas (I guess !).

THEOREM 31. *Let Γ be a rectifiable curve in $\mathbb{C}$ and let μ be a positive compactly supported measure with $supp\mu \subset \Gamma$. Assume that μ has linear growth:*

$$\mu(B) \leq diamB \text{ for every ball } B \subset \mathbb{C}.$$

Then, $c^2(\mu) \leq C \sum_{Q\in\Delta} \beta_\infty^\Gamma(Q)^2\mu(Q)$
(or equivalently, $c^2(\mu) \leq C \int_{\mathbb{C}}\int_0^{+\infty} \beta_\infty^\Gamma(x,t)^2 d\mu(x)\frac{dt}{t}$.)

In particular, by P. Jones'traveling salesman theorem, this implies that $c^2(\mu) \leq Cl(\Gamma) < +\infty$. From this and Frostman's lemma, we get

COROLLARY 32. *Let E be a compact set in the complex plane $\mathbb{C}$. Assume that $H^1(E) > 0$ and that E is contained in a rectifiable curve Γ. Then, there exists a positive Radon measure μ compactly supported on E with linear growth and finite Menger curvature.*

We say that an Ahlfors-regular set $E \subset \mathbb{C}$ satisfies the local curvature condition if there exists a constant $C > 0$ such that

$$\int\int\int_{(E\cap B)^3} c(x,y,z)^2 dH^1(x)dH^1(y)dH^1(z) \leq C\mathrm{diam}B$$

whenever B is a ball in $\mathbb{C}$. For instance, by theorem 30, a Lipschitz graph satisfies the local curvature condition.

From theorems 25 and 31, we get

THEOREM 33. *Let E be an Ahlfors-regular set in $\mathbb{C}$. If E is uniformly rectifiable, then E satisfies the local curvature condition.*

Remark. The estimate of $c^2(\mu)$ in theorem 31 can be improved if you assume that the curve Γ has some extra properties. For instance, if the measure μ is supported by an Ahlfors regular curve, then $c^2(\mu) \leq C\mu(\mathbb{C})$ where C depends on the regularity constant of the curve. In general, this estimate is not true (see [**88**]).

PROOF. We start with three technical lemmas.

LEMMA 34. *Let x, y, z and z' be four points in the complex plane $\mathbb{C}$ such that*

(i) *x, z, z' are collinear, and $z \in [x, z']$;*
(ii) $|y - z'| \leq |y - z|$.

Then, $c(x, y, z) \leq c(x, y, z')$.

Proof of lemma 34. First, Thales theorem gives

$$\frac{d(z, L_{xy})}{d(z', L_{xy})} = \frac{|x - z|}{|x - z'|}$$

where L_{xy} denotes the line passing through x and y.
Therefore,

$$\begin{aligned} c(x, y, z) &= \frac{2d(z, L_{xy})}{|x - z||y - z|} \\ &= \frac{2d(z', L_{xy})}{|x - z'||y - z|} \\ &\leq \frac{2d(z', L_{xy})}{|x - z'||y - z'|} = c(x, y, z'). \end{aligned}$$

Note that the last estimate follows from (ii).

The next result can be found in [**99**].

LEMMA 35. *Let x, y, z and z' be four distinct points in the complex plane $\mathbb{C}$ such that*

$$A^{-1}|y - z| \leq |y - z'| \leq A|y - z|$$

for some constant $A > 0$. Then,

$$|c(x, y, z) - c(x, y, z')| \leq (4 + 2A)\frac{|z - z'|}{|z - x||z - y|}.$$

Proof of lemma 35.

$$\begin{aligned}|c(x,y,z)-c(x,y,z')| &= \left|\frac{2d(z,L_{xy})}{|z-x||z-y|}-\frac{2d(z',L_{xy})}{|z'-x||z'-y|}\right| \\ &= 2\left|\frac{d(z,L_{xy})|z'-x||z'-y|-d(z',L_{xy})|z-x||z-y|}{|z-x||z-y||z'-x||z'-y|}\right| \\ &\le 2\frac{|d(z,L_{xy})-d(z',L_{xy})||z'-x||z'-y|}{|z-x||z-y||z'-x||z'-y|} \\ &\quad +2\frac{d(z',L_{xy})\,||z'-x||z'-y|-|z-x||z-y||}{|z-x||z-y||z'-x||z'-y|} \\ &= I+II\end{aligned}$$

Since $|d(z,L_{xy})-d(z',L_{x,y})|\le|z-z'|$, we get

$$I\le 2\frac{|z-z'|}{|z-x||z-y|}.$$

Furthermore,

$$\begin{aligned}||z'-x||z'-y|-|z-x||z-y|| &= |(|z'-x|-|z-x|)|z'-y|+(|z'-y|-|z-y|)|z-x|| \\ &\le |z'-z||z'-y|+|z'-z||z-x|.\end{aligned}$$

Thus, since $d(z',L_{xy})\le|z'-y|$ and $d(z',L_{xy})\le|z'-x|$, we get

$$\begin{aligned}II &\le 2|z'-z|\left(\frac{d(z',L_{xy})|z'-y|}{|z'-x||z'-y||z-x||z-y|}+\frac{d(z',L_{xy})|z'-x|}{|z'-x||z'-y||z-x||z-y|}\right) \\ &\le \frac{2|z'-z|}{|z-x||z-y|}+\frac{2|z'-z|}{|z-x||z'-y|} \\ &\le (2+2A)\frac{|z'-z|}{|z-x||z-y|}.\end{aligned}$$

Adding the inequalities obtained for I and II, we get the conclusion of lemma 35.

LEMMA 36. *Let x, y and z be three points of Γ such that*

(i) $|x-z|\le|x-y|$ *and* $|y-z|\le|x-y|$*;*

(ii) $\frac{|x-y|}{2^{n+1}}\le|x-z|\le\frac{|x-y|}{2^n}$ *for some* $n\in\mathbb{N}$*.*

Then,

$$c(x,y,z)\le\frac{C}{|x-y|}\sum_{j=0}^{n}\beta_\infty\left(x,3\frac{|x-y|}{2^j}\right).$$

Proof of lemma 36. Let ε_0 be small enough.

Assume first that there exists $k\in\{1,2,....,n\}$ such that $\beta_\infty\left(x,3\frac{|x-y|}{2^k}\right)\ge\varepsilon_0$. Therefore,

$$c(x,y,z)\le\frac{2}{|x-y|}\le\frac{2}{\varepsilon_0|x-y|}\beta_\infty\left(x,3\frac{|x-y|}{2^k}\right)\le\frac{C}{|x-y|}\sum_{j=0}^{n}\beta_\infty\left(x,3\frac{|x-y|}{2^j}\right).$$

Assume now that for any $j\in\{1,2,...,n\}$, $\beta_\infty\left(x,3\frac{|x-y|}{2^j}\right)\le\varepsilon_0$.

For any $k=1,...,n-1$, there exist points $\xi\in\Gamma$ such that $|x-\xi|=\frac{|x-y|}{2^k}$. Choose one of these points which we denote by ξ_k such that $|y - \xi_k|$ is minimal. Let D_k be the line passing through x and ξ_k, and denote by D_n the line passing through x and z (see figure 3).

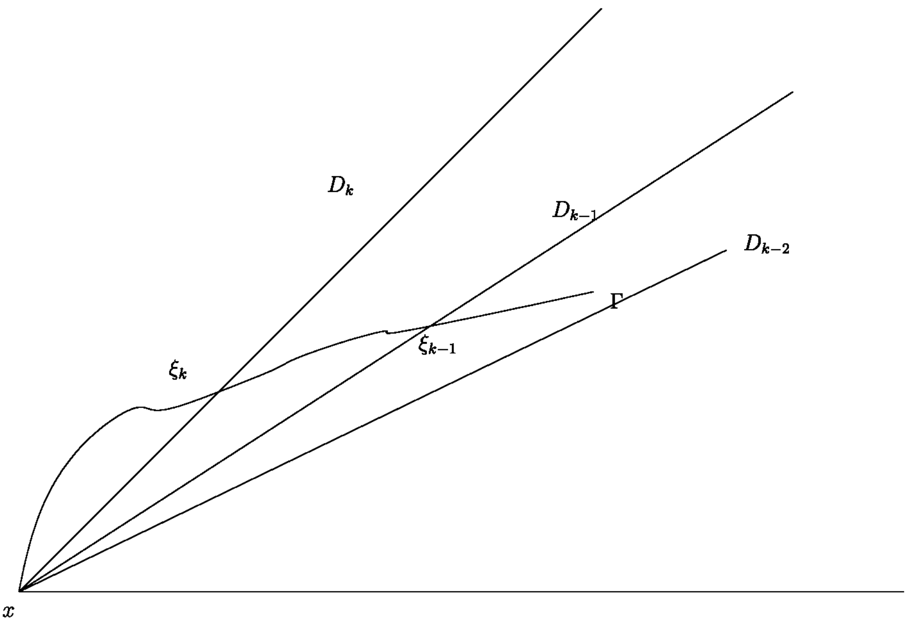

FIGURE 3.

Let L_k be a line minimizing $\beta_\infty\left(x, \frac{3|x-y|}{2^k}\right)$.

Then, since $d(x, L_k) \le 3\beta_\infty\left(x, \frac{3|x-y|}{2^k}\right)\frac{|x-y|}{2^k}$, $d(\xi_k, L_k) \le 3\beta_\infty\left(x, \frac{3|x-y|}{2^k}\right)\frac{|x-y|}{2^k}$ and $|x - \xi_k| = \frac{|x-y|}{2^k}$, we get

$$d(w, D_k) \le C\beta_\infty\left(x, \frac{3|x-y|}{2^k}\right)\frac{|x-y|}{2^k}$$

whenever $w \in \Gamma \cap D\left(x, \frac{3|x-y|}{2^k}\right)$. Moreover, by the definition of the betas, for any $w \in \Gamma \cap D\left(x, \frac{3|x-y|}{2^k}\right)$,

$$d(w, L_{k-1}) \le 3\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right)\frac{|x-y|}{2^{k-1}}$$

and hence,

$$\beta_\infty\left(x, \frac{3|x-y|}{2^k}\right) \le C\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right). \tag{17}$$

From all these remarks, we get that, for any $w \in \Gamma \cap D\left(x, \frac{3|x-y|}{2^k}\right)$,

$$d(w, D_k) \leq C\beta_\infty\left(x, \frac{3|x-y|}{2^k}\right)\frac{|x-y|}{2^k} \tag{18}$$

$$\leq C\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right)\frac{|x-y|}{2^{k-1}} \tag{19}$$

and

$$d(w, D_{k-1}) \leq C\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right)\frac{3|x-y|}{2^{k-1}}. \tag{20}$$

Thus,

$$\text{angle}(D_k, D_{k-1}) \leq C\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right). \tag{21}$$

To see this, consider a_k and b_k two points such that $|x-a_k| = |x-b_k| = \frac{|x-y|}{2^k}$, $a_k \in D_k$, $b_k \in D_{k-1}$ and a_k, b_k are chosen like on figure 4:

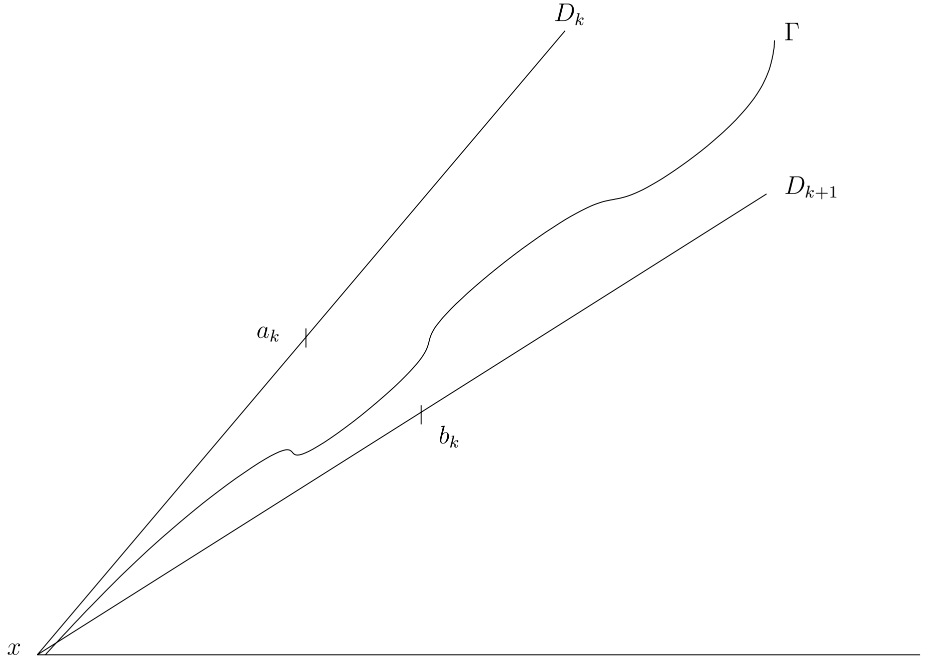

FIGURE 4.

Since Γ is connected, there exists $w_k \in \Gamma \cap D\left(x, \frac{3|x-y|}{2^k}\right)$ such that

$$|w_k - a_k| \leq C\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right)\frac{|x-y|}{2^{k-1}}(\text{from (19)}),$$

$$|w_k - b_k| \leq C\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right)\frac{|x-y|}{2^{k-1}}(\text{from(20)}).$$

Therefore, $|a_k - b_k| \leq C\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right)\frac{|x-y|}{2^{k-1}}$ and (21) follows easily.

For any $k = 1, ..., n-1$, let η_k be the point of the intersection of the line D_k and the circle centered at x of radius $|x-z|$ which is closer to y.

Hence, by (21), $|\eta_k - \eta_{k-1}| \leq C|x-z|\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right)$ (if ε_0 is small enough). By lemma 35, this yields

$$\begin{aligned} c(x,y,\eta_k) &\leq c(x,y,\eta_{k-1}) + C\frac{|\eta_k - \eta_{k-1}|}{|\eta_k - x||\eta_k - y|} \\ &\leq c(x,y,\eta_{k-1}) + C\frac{\beta_\infty(x, \frac{3|x-y|}{2^{k-1}})|x-z|}{|x-z||\eta_k - y|} \\ &\leq c(x,y,\eta_{k-1}) + \frac{C}{|x-y|}\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right). \end{aligned}$$

By induction, we get

$$c(x,y,z) \leq c(x,y,\eta_1) + C\sum_{k=2}^{n}\beta_\infty\left(x, \frac{3|x-y|}{2^{k-1}}\right). \tag{22}$$

If ε_0 is small enough, $|y-\xi_1| \leq |y-\eta_1|$. Hence, by lemma 34, $c(x,y,\eta_1) \leq c(x,y,\xi_1)$. Moreover, $c(x,y,\xi_1) \leq C\frac{\beta_\infty(x, 3|x-y|)}{|x-y|}$. Inequality (22) and these two remarks complete the proof of lemma 36.

We now go through the proof of the theorem. We start with some notations. Let Δ_j be the family of dyadic squares in $\mathbb{C}$ whose side length is 2^{-j}. Let $Q \in \Delta$. Denote by Q^* the square $3Q$. Write $P(Q) = \{R \in \Delta, Q \subset R\}$ and $F(Q) = \{R \in \Delta, R \subset Q\}$. If $x,\, y \in \Gamma$, we say that $(x,y) \in \tilde{Q}$ if $x,\, y \in Q^*$ and $\frac{1}{3}\text{diam}Q \leq |x-y| \leq 3\text{diam}Q$.
We now rewrite lemma 36 in terms of dyadic cubes.

LEMMA 37. *Assume that $Q \in \Delta$ and $R \in P(Q)$. Let x, y and z be three points in Γ such that $(x,z) \in Q^*$ and $(x,y) \in R^*$. Then,*

$$c(x,y,z) \leq \frac{C}{diamR}\sum_{Q\subset S\subset R}\beta_\infty(S).$$

Note that $c^2(\mu) \leq 3\tilde{c}^2(\mu)$ where $\tilde{c}^2(\mu) = \int\int\int_A c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z)$ and $A = \{(x,y,z) \in \Gamma^3; |x-z| \leq |x-y|, |y-z| \leq |x-y|\}$.
We now estimate $\tilde{c}^2(\mu)$.

$$\begin{aligned} \tilde{c}^2(\mu) &\leq \sum_{Q\in\Delta}\int_{(x,z)\in Q^*}\left(\sum_{R\in P(Q)}\int_{y\in R^*,(x,y)\in\tilde{R}} c(x,y,z)^2 d\mu(y)\right)d\mu(x)d\mu(z) \\ &\leq C\sum_{Q\in\Delta}\left(\sum_{R\in P(Q)}\frac{1}{(\text{diam}R)^2}\left(\sum_{Q\subset S\subset R}\beta_\infty(S)\right)^2\mu(R^*)\right)\mu(Q^*)^2. \end{aligned}$$

Let $\alpha \in]0, \frac{1}{2}[$. Then, for any $Q \in \Delta$ and any $R \in \Delta$ such that $Q \subset R$,

$$\begin{aligned}\frac{1}{(\text{diam}R)^2}\left(\sum_{Q\subset S\subset R}\beta_\infty(S)\right)^2 &= \frac{1}{(\text{diam}R)^{2-2\alpha}}\left(\sum_{Q\subset S\subset R}\frac{\beta_\infty(S)}{(\text{diam}R)^\alpha}\right)^2\\ &\leq \frac{1}{(\text{diam}R)^{2-2\alpha}}\left(\sum_{Q\subset S\subset R}\frac{\beta_\infty(S)}{(\text{diam}S)^\alpha}\right)^2\end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality, we get

$$\frac{1}{(\text{diam}R)^2}\left(\sum_{Q\subset S\subset R}\beta_\infty(S)\right)^2 \leq \frac{C}{(\text{diam}R)^{2-2\alpha}}\left(\sum_{Q\subset S\subset R}\beta_\infty(S)^2\right)\frac{1}{(\text{diam}Q)^{2\alpha}}.$$

Using this estimate and Fubini theorem, we get

$$\tilde{c}^2(\mu) \leq C\sum_{S\in\Delta}\beta_\infty(S)^2\left(\sum_{Q\in F(S)}\frac{\mu(Q^*)^2}{(\text{diam}Q)^{2\alpha}}\right)\left(\sum_{R\in P(S)}\frac{\mu(R^*)}{(\text{diam}R)^{2-2\alpha}}\right).$$

Moreover,

$$\sum_{R\in P(S)}\frac{\mu(R^*)}{(\text{diam}R)^{2-2\alpha}} \leq C\sum_{R\in P(S)}\frac{1}{(\text{diam}R)^{1-2\alpha}} \leq \frac{C}{(\text{diam}S)^{1-2\alpha}},$$

and if we assume that $S \in \Delta_j$,

$$\begin{aligned}\sum_{Q\in F(S)}\frac{\mu(Q^*)^2}{(\text{diam}Q)^{2\alpha}} &= \sum_{k=j}^{+\infty}\sum_{Q\in\Delta_k\cap F(S)}\frac{\mu(Q^*)^2}{(\text{diam}Q)^{2\alpha}}\\ &\leq C\sum_{k=j}^{+\infty}\sum_{Q\in\Delta_k\cap F(S)}(\text{diam}Q)^{1-2\alpha}\mu(Q^*)\\ &\leq C\sum_{k=j}^{+\infty}(2^{-k})^{1-2\alpha}\mu(S)\\ &\leq C(\text{diam}S)^{1-2\alpha}\mu(S).\end{aligned}$$

Therefore, $\tilde{c}^2(\mu) \leq C\sum_{S\in\Delta}\beta_\infty(S)^2\mu(S)$ and the proof of the theorem is complete. □

Conversely, if the set is Ahlfors-regular, we can control the betas by the Menger curvature.

THEOREM 38. *Let $E \subset \mathbb{C}$ be a Ahlfors-regular set (with dimension 1). Then, for all balls B of $\mathbb{C}$,*

$$\int_{E\cap B}\int_0^{diam\ B}\beta_\infty^E(x,t)^2 dH^1(x)\frac{dt}{t} \leq C\int\int\int_{(E\cap CB)^3}c(x,y,z)^2 dH^1(x)dH^1(y)dH^1(z)$$

where $C > 0$ depends only on the regularity constant of E.

We should emphasize that the betas in theorem 38 correspond to the set E, whereas the betas in theorem 31 correspond to the curve Γ.

From theorems 25, 33 and 38, we get a complete characterization of uniformly rectifiable sets in terms of Menger curvature.

THEOREM 39. *Let $E \subset C$ be a Ahlfors-regular set (with dimension 1). Then, E is uniformly rectifiable if and only if E satisfies the local curvature condition.*

We now prove theorem 38.

PROOF. Since E is Ahlfors-regular (with dimension 1), there exists a family $\Delta(E)$ of "dyadic cubes" associated to E (see proposition 24). We claim that the theorem 38 will follow from the

PROPOSITION 40. *Let $Q \in \Delta(E)$. Then,*

$$\sum_{S\subset Q, S\in\Delta(E)} \beta_\infty(S)^2 diam S \leq C \int\int\int_{(E\cap Q)^3} c(x,y,y)^2 dH^1(x)dH^1(y)dH^1(z).$$

To see this, consider $B = B(x,R)$. First, we claim that it is enough to consider the case $x \in E$ and $R \in (0, \mathrm{diam} E)$. Indeed, if $E \cap B = \emptyset$, then the integral in the right hand side of the inequality of theorem 38 is zero, and if $E \cap B \neq \emptyset$, then there exists $x_0 \in E$ such that $B \subset B(x_0, \mathrm{diam} B)$.
From now on, we assume that $x \in E$ and $R \in (0, \mathrm{diam} E)$. Let $j \in \mathbb{Z}$ such that $2^{-j} \leq R < 2^{-j+1}$. Then, there exist "dyadic cubes" $Q_i \in \Delta_j(E)$, $i = 1, 2, .., N$, such that $B \subset \bigcup_{i=1}^N Q_i$. Recall that $C^{-1}2^{-j} \leq \mathrm{diam} Q_i \leq C2^{-j}$ and $C^{-1}2^{-j} \leq H^1(Q_i) \leq C2^{-j}$ where $C > 0$ depends only on the regularity constant C_0 of E. From this, we get $\bigcup_{i=1}^N Q_i \subset B(x, CR)$ and therefore there exists N_0 depending only on the regularity constant of E such that $N \leq N_0$. Thus,

$$\begin{aligned}\int_0^R \int_{E\cap B(x,R)} \beta_\infty(y,t)^2 dH^1(y)\frac{dt}{t} &\leq C\sum_{i=1}^N \sum_{S\subset Q_i} \beta_\infty(S)^2 \mathrm{diam} S \\ &\leq C\sum_{i=1}^N \int_{Q_i}\int_{Q_i}\int_{Q_i} c(x_1,x_2,x_3)^2 dH^1(x_1)dH^1(x_2)dH^1(x_3) \\ &\leq C\int\int\int_{(E\cap B(x,CR))^3} c(x_1,x_2,x_3)^2 dH^1(x_1)dH^1(x_2)dH^1(x_3)\end{aligned}$$

where C depends only on the regularity constant of E.

We now prove proposition 40 and for this, we start with some notations. Write $\mu = H^1_{|E}$ and denote by $\tilde{\mu}$ the normalization of μ, that is

$$\int_A g d\tilde{\mu} = \frac{1}{\mu(A)} \int_A g d\mu.$$

Of course, $\tilde{\mu}$ is not a measure !
If z_1, z_2,..., z_n are n pairwise distinct points in $\mathbb{C}$, write

$$\beta(z_1, z_2, ..., z_n) = \frac{\omega(z_1, z_2, .., z_n)}{\max_{i\neq j} |z_i - z_j|}$$

where $\omega(z_1, z_2, .., z_n)$ is the width of a (infinite) strip of smallest possible width which contains z_1, z_2,..., z_n.

Fix $Q \in \Delta(E)$. Recall (see proposition 24) that, by construction, there exists a ball $B = B(x,R)$ such that $E \cap B \subset Q$ where $x \in E$ and $R = C^{-1}\text{diam}Q$. Then, we can find z_1, $z_2 \in E \cap B$ such that $|z_1 - z_2| \geq C^{-1}R$ (where $C > 0$ depends only on the regularity constant of E). Indeed, fix $z_1 \in E \cap B$ and assume that the conclusion does not hold, that is $|z_1 - z| \leq C^{-1}R$ whenever $z \in E \cap B$. Thus, $E \cap (B \setminus B(z_1, C^{-1}R)) = \emptyset$ and therefore, $\mu(B) = \mu(B(z_1, C^{-1}R))$. This yields

$$C_0^{-1}R \leq \mu(B) = \mu(B(z_1, C^{-1}R)) \leq C_0 C^{-1}R.$$

But, this estimate is impossible if $C >> C_0^2$ (here, C_0 is the regularity constant of E).

Consider now $z_0 \in E \cap B$ such that

$$\sup_{z \in E \cap B} \frac{d(z, L_{z_1 z_2})}{R} = \frac{d(z_0, L_{z_1 z_2})}{R}$$

where $L_{z_1 z_2}$ is the line of $\mathbb{C}$ passing through z_1 and z_2. Note that $\beta_\infty(Q) \leq C\dfrac{d(z_0, L_{z_1 z_2})}{R} \leq C\beta(z_0, z_1, z_2)$.

Claim: There exist $\lambda \in (0,1)$ (depending only on the regularity constant of E) and two collections of balls (B_n), $(\tilde{B}_n)$ centered on E such that

$$C^{-1}\lambda^n R \leq d(B_n, \tilde{B}_n) \leq C\lambda^n R$$

$$C^{-1}\lambda^n R \leq \text{diam}B_n = \text{diam}\tilde{B}_n \leq C\lambda^n R$$

$$C^{-1}\lambda^n R \leq d(z_0, B_n) \leq C\lambda^n R$$

$$C^{-1}\lambda^n R \leq d(z_0, \tilde{B}_n) \leq C\lambda^n R$$

$$C^{-1}\lambda^n R \leq d(B_n, B_{n+1}) \leq C\lambda^n R$$

$$C^{-1}\lambda^n R \leq d(\tilde{B}_n, \tilde{B}_{n+1}) \leq C\lambda^n R$$

where $C > 0$ depends only on the regularity constant of E (see figure 5).

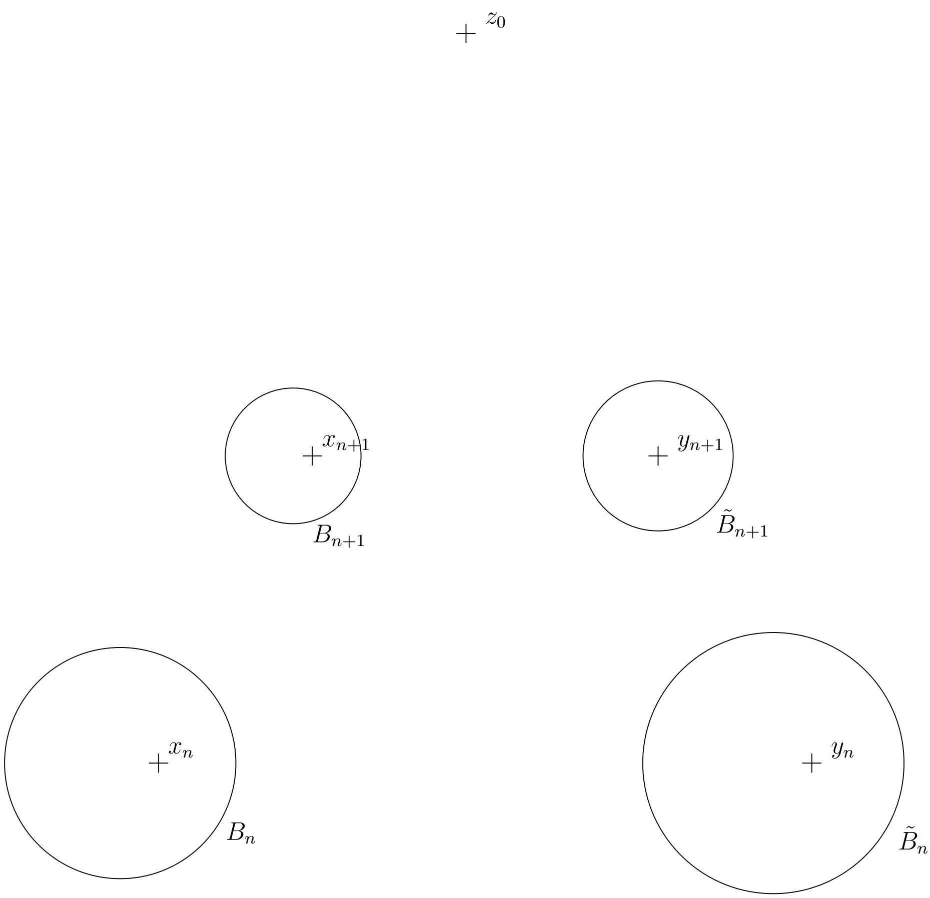

FIGURE 5.

Proof of the claim: We start with an easy lemma.

LEMMA 41. *Let $\xi \in E$ and let $r \in (0, diamE)$. Then, there exists $\delta \in (0,1)$ (depending only on the regularity constant of E) such that*

$$E \cap (B(\xi, r) \setminus B(\xi, \delta r)) \neq \emptyset.$$

PROOF. Assume that $E \cap (B(\xi, r) \setminus B(\xi, \delta r)) = \emptyset$. Then, $\mu(B(\xi, r)) = \mu(B(\xi, \delta r))$. Thus, since E is Ahlfors-regular (with constant C_0),

$$C_0^{-1} r \leq \mu(B(\xi, r)) = \mu(B(\xi, \delta r)) \leq C_0 \delta r.$$

From this, we get $\delta \geq C_0^{-2}$. Therefore, if $\delta < C_0^{-2}$,

$$E \cap (B(\xi, r) \setminus B(\xi, \delta r)) \neq \emptyset.$$

□

Fix now δ given by the lemma and for $n \in \mathbb{N}$, write

$$A_n = B(z_0, \delta^{2n} R) \setminus B(z_0, \delta^{2n+2} R)$$
$$C_n = B(z_0, \delta^{2n+1} R) \setminus B(z_0, \delta^{2n+2} R).$$

Note that $C_n \subset A_n$ and $E \cap C_n \neq \emptyset$.
Let $\xi_n \in E \cap C_n$. Then,

$$B\left(\xi_n, \frac{\delta^{2n+2}R}{1000}\right) \subset B\left(z_0, \frac{3\delta^{2n+1}R}{2}\right) \setminus B\left(z_0, \frac{\delta^{2n+2}R}{10}\right).$$

By using the lemma again, there exists $\zeta_n \in E \cap \left(B\left(\xi_n, \frac{\delta^{2n+2}R}{1000}\right) \setminus B\left(\xi_n, \frac{\delta^{2n+3}R}{1000}\right)\right)$.
Set $\lambda = \delta^2$ and

$$B_n = B(\xi_n, \lambda^{n+2}R)$$
$$\tilde{B}_n = B(\zeta_n, \lambda^{n+2}R).$$

If $\delta > 0$ is small enough, the balls (B_n) and $(\tilde{B}_n)$ satisfy the conclusion of the claim.

Step 0 : Up to some modifications in the construction above, we can assume that the sets B_0, $\tilde{B}_0$ and $S = \bigcup_{j=0}^{2} B(z_j, \delta R)$ are pairwise disjoint.
Define

$$\begin{aligned} F_0(x,y) &= \int_S c(x,y,z)^2 d\tilde{\mu}(z), \\ G_0 &= \int_{B_0}\int_{\tilde{B}_0} F_0(x,y) d\tilde{\mu}(x) d\tilde{\mu}(y). \end{aligned}$$

By Tchebychev inequality, there exist $x_0 \in B_0$ and $y_0 \in \tilde{B}_0$ such that $F_0(x_0, y_0) \leq CG_0$. Note that $\beta(z_0, z_1, z_2) \leq C \sum_{j=0}^{2} \beta(z_j, x_0, y_0)$ (where C depends on λ and the regularity constant of E). Thus, even if this means changing the labelling of the points z_0, z_1, z_2, we can assume that $\beta(x_0, y_0, z_0) \geq C^{-1}\beta(z_0, z_1, z_2)$.

Our goal is to choose "good" points $x_n \in B_n$, $y_n \in \tilde{B}_n$. We now explain what means "good". For this, assume that these points have been chosen. Then,

$$\beta_\infty(Q) \leq C\beta(z_0, z_1, z_2) \leq \frac{C}{R}\sum_n \omega(x_n, y_n, x_{n+1}, y_{n+1}).$$

But, $\omega(x_n, y_n, x_{n+1}, y_{n+1})$ is comparable to $\beta(x_n, y_n, x_{n+1}, y_{n+1})\lambda^n R$, and if we set $c_n = c(x_n, y_n, x_{n+1}) + c(x_n, y_n, y_{n+1})$, c_n is comparable to $\beta(x_n, y_n, x_{n+1}, y_{n+1})\lambda^{-n}R^{-1}$. From this, we get

$$\beta_\infty(Q) \leq \frac{C}{R}\sum_n \omega(x_n, y_n, x_{n+1}, y_{n+1}) \leq \frac{C}{R}\sum_n c_n(\lambda^n R)^2.$$

Since we would like to control $\beta_\infty(Q)$ by a triple integral of Menger curvature, a natural idea is to choose the points (x_n) and (y_n) such that

$$c_n^2 \leq C\int_{B_n}\int_{\tilde{B}_n}\int_{B_{n+1}\cup\tilde{B}_{n+1}} c(x,y,z)^2 d\tilde{\mu}(x) d\tilde{\mu}(y) d\tilde{\mu}(z).$$

Step n: Assume that we have chosen $x_n \in B_n$ and $y_n \in \tilde{B}_n$ such that

(i) $F(x_n, y_n) \leq C_1 G_n$,

(ii) $c(x_{n-1}, y_{n-1}, x_n)^2 \leq C_2 G_{n-1}$ and $c(x_{n-1}, y_{n-1}, y_n)^2 \leq C_2 G_{n-1}$

where, as previously,

$$\begin{aligned} F_n(x,y) &= \int_{B_{n+1}\cup\tilde{B}_{n+1}} c(x,y,z)^2 d\tilde{\mu}(z), \\ G_n &= \int_{B_n}\int_{\tilde{B}_n} F_n(x,y)d\tilde{\mu}(x)d\tilde{\mu}(y). \end{aligned}$$

By using the Tchebychev inequality, the definitions of F_n and G_n, the choice of x_n and y_n, we get

$$\begin{aligned} \tilde{\mu}(\{z\in B_{n+1}\cup\tilde{B}_{n+1}; c(x_n,y_n,z)^2\geq C_2G_n\}) &< \frac{C_1}{C_2} \\ \tilde{\mu}\times\tilde{\mu}(\{(x,y)\in B_{n+1}\times\tilde{B}_{n+1}; F_{n+1}(x,y)\geq C_1G_{n+1}\}) &< \frac{1}{C_1} \end{aligned}$$

By using these estimates, we can easily construct $x_{n+1}\in B_{n+1}$ and $y_{n+1}\in\tilde{B}_{n+1}$ which satisfy the induction properties (i) and (ii) above.
Thus,

$$\beta_\infty(Q)\leq\frac{C}{R}\sum_n c_n(\lambda^nR)^2$$

where

$$c_n\leq C\int_{B_n}\int_{\tilde{B}_n}\int_{B_{n+1}\cup\tilde{B}_{n+1}} c(x,y,z)^2d\tilde{\mu}(x)d\tilde{\mu}(y)d\tilde{\mu}(z).$$

Then, by Hölder inequality, we get

$$\begin{aligned} \beta_\infty(Q)^2 &\leq C\left(\sum c_n\lambda^{2n}R\right)^2 \\ &\leq C\left(\sum c_n^2(\lambda^nR)^3(\lambda^nR)^{\frac{1}{2}}\right)\sum\lambda^{\frac{n}{2}}(R)^{\frac{-3}{2}}. \end{aligned}$$

Since $\sum\lambda^{\frac{n}{2}}<+\infty$, this yields

$$\beta_\infty(Q)^2\leq CR^{\frac{-3}{2}}\sum_n c_n^2(\lambda^nR)^3(\lambda^nR)^{\frac{1}{2}}.$$

Moreover, since $\mu(B_n)$ and $\mu(\tilde{B}_n)$ are comparable to λ^nR,

$$c_n^2(\lambda^nR)^3\leq C\int_{B_n}\int_{\tilde{B}_n}\int_{B_{n+1}\cup\tilde{B}_{n+1}} c(x,y,z)^2d\mu(x)d\mu(y)d\mu(z).$$

Thus,

$$\beta_\infty(Q)^2\mathrm{diam}Q\leq CR^{-\frac{1}{2}}\sum_n\left(\int_{B_n}\int_{\tilde{B}_n}\int_{B_{n+1}\cup\tilde{B}_{n+1}} c(x,y,z)^2d\mu(x)d\mu(y)d\mu(z)(\lambda^nR)^{\frac{1}{2}}\right)$$

and therefore

$$\beta_\infty(Q)^2\mathrm{diam}Q\leq C\sum_{P\subset Q,P\in\Delta(E)}\int_{P^*} c(x,y,z)^2d\mu(x)d\mu(y)d\mu(z)\left(\frac{\mathrm{diam}P}{\mathrm{diam}Q}\right)^{\frac{1}{2}}$$

where P^* is the set of triples of points $(x_1,x_2,x_3)\in(3P)^3$ such that $C^{-1}\mathrm{diam}P\leq|x_i-x_j|\leq 3\ \mathrm{diam}P$ for $i\neq j$.

Fix now $Q \in \Delta(E)$. By Fubini theorem, we get (the sum over S and P are for $S, P \in \Delta(E)$)

$$\begin{aligned}\sum_{S\subset Q} \beta_\infty(S)^2 \mathrm{diam} S &\leq C \sum_{S\subset Q}\sum_{P\subset S} \int_{P^*} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z) \left(\frac{\mathrm{diam} P}{\mathrm{diam} S}\right)^{\frac{1}{2}} \\ &\leq C \sum_{P\subset Q} \int_{P^*} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z) \\ &\leq C \int_Q\int_Q\int_Q c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z).\end{aligned}$$

and the proof is complete.

□

4. Menger curvature and Cantor type sets

We start with a general result of P. Mattila [**66**].

THEOREM 42. *Let $h : [0, +\infty] \to [0, \infty]$ be a non decreasing function such that $\int_0^{+\infty} r^{-3}h^2(r)dr < \infty$ and let μ be a finite Borel measure in $\mathbb{C}$ such that, for every ball B of radius $r > 0$, $\mu(B) \leq Ch(r)$. Then,*

$$c^2(\mu) \leq C\mu(\mathbb{C}) \int_0^{+\infty} r^{-3}h^2(r)dr.$$

PROOF. We follow the proof given by P. Mattila in [**66**].
We start with some observations. Set $\mu_r(y) = \mu(B(y,r))$.
Then, $\left(\frac{h(r)}{r}\right)^2 \leq 8\int_r^{2r} t^{-3}h^2(t)dt$, hence $\lim_{r\to 0} \frac{h(r)}{r} = 0$.
From this, we get

$$\lim_{r\to 0} \frac{\mu_r(y)}{r} = 0 \text{ since } \frac{\mu_r(y)}{r} \leq \frac{h(r)}{r}. \tag{23}$$

$$\lim_{r\to +\infty} \frac{\mu_r(y)}{r} = 0 \text{ since } \frac{\mu_r(y)}{r} \leq \frac{\mu(\mathbb{C})}{r}. \tag{24}$$

If x, y and z are three points of $A = \{(x,y,z); |x-y| \leq |x-z| \text{ and } |x-y| \leq |y-z|\}$, then

$$c(x,y,z) = \frac{2d(z, L_{x,y})}{|x-z||y-z|} \leq 2\frac{1}{|y-z|}.$$

Therefore, we get

$$\begin{aligned}
c^2(\mu) &\leq 3\int_A\int_A\int_A c^2(x,y,z)d\mu(x)d\mu(y)d\mu(z)\\
&\leq 12\int_A\int_A\int_A |y-z|^{-2}d\mu(x)d\mu(y)d\mu(z)\\
&\leq \int_A\int_A\int_{B(y,|y-z|)} |y-z|^{-2}d\mu(x)d\mu(y)d\mu(z)\\
&= 12\int\int \frac{\mu(B(y,|y-z|))}{|y-z|^2}d\mu(y)d\mu(z)\\
&= 12\int\int_0^\infty \frac{\mu_y(r)}{r^2}d\mu_y(r)d\mu(y) \text{ (in the Riemann-Stieltjes sense)}
\end{aligned}$$

By integrating by parts, and by using (23) and (24), we get

$$c^2(\mu) \leq 12\int\int \frac{\mu_y(r)^2}{r^3}drd\mu(y) \leq 12\mu(\mathbb{C})\int_0^\infty \frac{h^2(r)}{r^3}dr.$$

□

By Frostman's lemma and theorem 42, we easily get

Corollary 43. *Let $E \subset \mathbb{C}$ with $dim_H(E) > 1$. Then there exists a positive finite Radon measure μ compactly supported on E with linear growth and finite Menger curvature.*

Let (λ_n) be a non-increasing sequence of real numbers with $0 < \lambda < \frac{1}{2}$. We associate a generalized four corners Cantor set E to this sequence in the following way.
Let K_0 be the unit interval $[0,1]$ and let $K_1 = [0,\lambda_1] \cup [1-\lambda_1, 1]$. Assume that K_{n-1} have been constructed. The set K_n is obtained from K_{n-1} by replacing each component of K_{n-1} by the two endmost intervals of length $\lambda_1......\lambda_n$. Set $K = \cap_n K_n$ and then, $E = K \times K$. Note that $E = \bigcap_{n=1}^{\infty}\bigcup_{j=1}^{4^n} Q_n^j$, where Q_n^j is a square of side length $\sigma_n = \lambda_1....\lambda_n$. The Cantor set constructed in the first chapter corresponds to the constant sequence $\lambda_n = \frac{1}{4}$.
Let $h : [0,+\infty) \to [0,+\infty)$ be a non-decreasing function such that $h(0) = 0$ and $h(\sigma_n) = 4^{-n}$ (for $n = 1,2,..$) and let Λ_h be the Hausdorff measure associated to the gauge h (see chapter 1).

Theorem 44. *With the same notations as above, we have*
(i) $0 < \Lambda_h(E) < +\infty$ and there exists $C > 1$ such that

$$Ch(r) \leq \Lambda_h(E \cap B(x,r)) \leq Ch(r)$$

whenever $x \in E$ and $0 < r < 2$.
(ii) If $\sum_n \left(\frac{4^{-n}}{\sigma_n}\right)^2 < +\infty$, then $c^2(\Lambda_h) < +\infty$.

The proof of (i) is left to the reader. The part (ii) follows from P. Mattila's result by noting that the sum $\sum_n \left(\frac{4^{-n}}{\sigma_n}\right)^2$ is comparable to $\int_0^{+\infty} r^{-3}h^2(r)dr$.

5. P. Jones' construction of "good" measures supported on continua

In this section, we will prove that any continuum E of the complex plane $\mathbb{C}$ supports a positive Radon measure with linear growth and finite Menger curvature. As for theorems 31 and 38, this result has been announced by P. Jones who never published his proof. The construction we will give is widely inspired by handwritten notes of G. David (which are based on conversations with P. Jones). Recall that a continuum $E \subset \mathbb{C}$ is a compact connected set.

THEOREM 45. *There exists $C_0 > 0$ such that, for any continuum $E \subset \mathbb{C}$, there exists a positive Radon measure μ supported on E which satisfies*

(i) $\mu(E) \geq C_0^{-1} diam E$;
(ii) $\mu(B(x,r)) \leq C_0 r$ *whenever* $x \in \mathbb{C}$ *and* $r > 0$;
(iii) $c^2(\mu) \leq C_0 diam E$.

PROOF. The strategy of the proof is the following. By using theorem 22 and a stopping time argument involving the beta numbers, we associate to each ball B centered on E an Ahlfors regular curve Γ_B which contains a large part of $E \cap B$ or approximates a large part of $E \cap B$. From this, we construct a sequence of measures μ_j which converges weakly to a measure μ supported on E. Finally, we check that this measure satisfies (i), (ii) and (iii). Here, we use the fact that the restriction of H^1 to any Ahlfors regular curve has linear growth and satisfies the local curvature condition (by theorem 33).

Let $M > 0$ big enough.

Construction of μ (Step 1)

Let $B = B(x_B, R_B)$ be a ball centered on E and with radius $R_B \leq \text{diam}E$. Let K_B be the connected component of $E \cap \overline{B}$ which contains x_B. We leave to the reader to check that K_B is a continuum with $\text{diam}K_B \geq R_B$.
For any $x \in K_B$, set

$$t_B(x) = \inf\left\{t \in (0, R_B); \int_t^{R_B} \beta_\infty^{K_B}(x,t)^2 \frac{dt}{t} \leq M\right\}.$$

Therefore, $t_B(x) = 0$ if and only if $\displaystyle\int_0^{R_B} \beta_\infty^{K_B}(x,t)^2 \frac{dt}{t} \leq M$.

Write $Z(B) = \overline{\{x \in K_B; t_B(x) = 0\}}$.
By easy computations, we get, for any $x \in Z(B)$,

$$\int_0^{R_B} \beta_\infty^{Z(B)}(x,t)^2 \frac{dt}{t} \leq \int_0^{R_B} \beta_\infty^{K_B}(x,t)^2 \frac{dt}{t} \leq 4M.$$

We say that $B \in \mathcal{G}$ (respectively $B \in \mathcal{B}$) if $H^1(Z(B)) \geq \dfrac{1}{100} R_B$ (respectively $H^1(Z(B)) < \dfrac{1}{100} R_B$).

Assume first that $B \in \mathcal{G}$. Since $\displaystyle\int_0^{R_B} \beta_\infty^{Z(B)}(x,t)^2 \frac{dt}{t} \leq 4M$ for any $x \in Z(B)$, theorem 22 (see the remark after theorem 22) implies that $Z(B)$ is contained in a Ahlfors-regular curve Γ_B (whose regularity constant depends only on M). Set $\mu_B = H^1_{\lceil Z(B)}$. Then,

(i) $\mu_B(\mathbb{C}) = H^1(Z(B)) \geq \dfrac{R_B}{100}$;
(ii) $\mu_B(B(x,r)) \leq Cr$ whenever $x \in \mathbb{C}$ and $r > 0$;
(iii) $c^2(\mu_B) \leq CR_B$ by theorem 39,

where the constant $C > 0$ depends only on M. We should emphasize that all the constants C in this proof depend on M and on nothing else.

Assume now that $B \in \mathcal{B}$. Set $G(B) = (E \cap B) \setminus Z(B)$. Then, by a standard covering lemma (see [**54**] page 2), there exists a countable set $X(B) \subset G(B)$ such that $G(B) \subset \cup_{x\in X(B)} B(x, 10t_B(x))$ and the balls $B(x, 2t_B(x))$, $x \in X(B)$, are pairwise disjoint. If M is big enough, then $t_B(x) \leq \dfrac{R_B}{10^4}$, since the beta numbers are always less than 1.
We claim that

$$\sum_{x\in X(B)} t_B(x) \geq 10R_B. \tag{25}$$

Indeed, for $x \in X(B)$, set

$$H(x) = \partial B(x, 2t_B(x)) \cup \partial B(x, 10t_B(x)) \cup D_x$$

where D_x is any diameter of $\partial B(x, 10t_B(x))$.
Then, $H(x)$ is connected and $H^1(H(x)) \leq 100t_B(x)$. Set $E(B) = Z(B) \cup \left(\cup_{x\in X(B)} H(x)\right)$. We leave to the reader to check that $E(B)$ is compact and connected (recall that E is a continuum !!!). Moreover, since $B \in \mathcal{B}$,

$$\begin{aligned} H^1(E(B)) &\leq H^1(Z(B)) + 100 \sum_{x\in X(B)} t_B(x) \\ &\leq \frac{1}{100} R_B + 100 \sum_{x\in X(B)} t_B(x). \end{aligned}$$

To conclude, we need a lower bound and for this, we will use P. Jones' traveling salesman theorem.
Let $y \in E(B) \setminus Z(B)$. Then, there exists $x \in X(B)$ such that $y \in H(x)$. Let $x_0 \in X(B)$ such that

$$t_B(x_0) = \sup\{t_B(x); y \in H(x)\}.$$

We would like to prove that, if $t \geq 10^3 t_B(x_0)$, then

$$\beta_\infty^{E(B)}(y,t) \geq 10^{-2} \beta_\infty^E(x_0, \frac{t}{100}). \tag{26}$$

For this, let L be a line minimizing $\beta_\infty^{E(B)}(y,t)$, that is such that

$$\frac{d(z,L)}{t} \leq \beta_\infty^{E(B)}(y,t) \text{ whenever } z \in B(y,t) \cap E(B). \tag{27}$$

Fix now $z \in B\left(x_0, \frac{t}{100}\right) \cap E$. Then,

$$\begin{aligned} |z-y| &\leq |z-x_0| + |x_0-y| \\ &\leq \frac{t}{100} + 20t_B(x_0) \\ &\leq \frac{3t}{100} \text{ (by the choice of } t). \end{aligned}$$

Therefore, $z \in B\left(y, \frac{3t}{100}\right)$. We divide now the proof of (26) into three cases.

Case 1. $z \in Z(B)$.
Since $z \in E(B) \cap B\left(y, \frac{3t}{100}\right) \subset E(B) \cap B(y,t)$ we get from (27),

$$\frac{d(z,L)}{t} \leq \beta_\infty^{E(B)}(y,t). \tag{28}$$

From now on, we assume that $z \notin Z(B)$. Therefore, by construction, there exists $x \in X(B)$ such that $z \in B(x, 10t_B(x))$.

Case 2. $\overline{B(x,10t_B(x))} \subset B(y,t)$.
Thus, $H(x) \subset B(y,t) \cap E(B)$ and

$$\frac{d(z,L)}{t} \leq \beta_\infty^{E(B)}(y,t). \tag{29}$$

Case 3. $\overline{B(x,10t_B(x))}$ is not contained in $B(y,t)$.
Note that, since $z \in B(x,10t_B(x)) \cap B\left(y, \frac{3t}{100}\right)$, $20t_B(x) \geq \frac{97}{100}t$ and then

$$30t_B(x) \geq t. \tag{30}$$

Since $t \geq 10^3 t_B(x_0)$, we get $t_B(x) \geq t_B(x_0)$ and thus, by the choice of x_0, $y \notin B(x,10t_B(x))$. But, $z \in B(x, t_B(x)) \cap B(y, \frac{3t}{100})$. Hence, $\frac{3t}{100} \geq |y-z| \geq 10t_B(x)$. This implies $t > 30t_B(x)$ and this inequality contradicts (30). Therefore, this case is impossible. Thus, (26) follows from (28) and (29).

By integrating (26), we get

$$\begin{aligned} \int_{10^3 t_B(x_0)}^{R_B} \beta_\infty^{E(B)}(x,t)^2 \frac{dt}{t} &\geq 10^{-4} \int_{10^3 t_B(x_0)}^{R_B} \beta_\infty^{E\cap B}(x_0,t)^2 \frac{dt}{t} \\ &\geq 10^{-4}M - \log 10 \text{ (since } x_0 \notin Z(B)) \\ &\geq 10^{-5}M \text{ (if } M \text{ is big enough)}. \end{aligned}$$

Thus, for any $y \in E(B) \setminus Z(B)$,

$$\int_0^{R_B} \beta_\infty^{E(B)}(y,t)^2 \frac{dt}{t} \geq 10^{-5}M. \tag{31}$$

We now prove (25) by contradiction. For this, assume that $\sum_{x\in X(B)} t_B(x) \leq 10R_B$. Then, $H^1(E(B)) \leq (10^{-1}+10^3)R_B$. Thus, by theorem 18,

$$\int_{x\in E(B)} \int_0^{R_B} \beta_\infty^{E(B)}(x,t)^2 dH^1(x) \frac{dt}{t} \leq CR_B. \tag{32}$$

On the other hand, since $E(B)$ is connected,

$$H^1(E(B)) \geq \mathrm{diam} E(B) \geq R_B.$$

Since $H^1(Z(B)) \leq \frac{1}{100} R_B$, we get $H^1(E(B) \setminus Z(B)) \geq \frac{R_B}{2}$. From this and (31), we get

$$\begin{aligned}\int_{x \in E(B)\setminus Z(B)} \int_0^{R_B} \beta_\infty^{E(B)}(x,t)^2 dH^1(x) \frac{dt}{t} &\geq 10^{-5} M H^1(E(B) \setminus Z(B)) \\ &\geq 10^{-5} M \frac{R_B}{2}.\end{aligned}$$

This contradicts (32) if M is big enough, and the proof of (25) is complete. For any $x \in K_B \cup X(B)$, $\int_0^{R_B} \beta_\infty^{K_B \cup X(B)}(x,t)^2 \frac{dt}{t} \leq 100M$. Therefore, by theorem 22, $K_B \cup X(B)$ is contained in a Ahlfors regular curve denoted by Γ_B. Since the balls $B(x, t_B(x))$, $x \in X(B)$, are pairwise disjoint and $H^1(\Gamma_B \cap B(x, t_B(x)))$, $x \in X(B)$, is comparable to $t_B(x)$, we can assume that the curve Γ_B contains a cross $G(x)$ for any $x \in X(B)$. This cross $G(x)$ is just the union of two perpendicular segments centered on x and whose diameter is $2t_B(x)$. We leave to the reader to check the details.
Let μ_B be the restriction to $\cup_{x \in X(B)} G(x)$ of $H^1_{\lceil \Gamma_B}$.

Construction of μ (Step 2)

Without loss of generality, we can assume that $0 \in E$, $E \subset D(0,1)$ and $\mathrm{diam} E \geq \frac{1}{10}$. We first construct by induction a collection of balls $\mathcal{F} = \cup_{j \in \mathbb{N}} \mathcal{F}_j$ as follows:
- $\mathcal{F}_0 = \{D(0,1)\}$.
- If we assume that $\mathcal{F}_n$ has been build, we set

$$\mathcal{F}_{n+1} = \{B(x, t_B(x)), x \in X(B) \text{ and } B \in \mathcal{F}_n \cap \mathcal{B}\}.$$

Remarks. 1) If every $B \in \mathcal{F}_n$ is also in $\mathcal{G}$, then $\mathcal{F}_{n+1} = \emptyset$.
2) Let B_1, B_2 be two balls of $\mathcal{F}_k$. If B and B' are two balls of $\mathcal{F}_{k+1}$ which are contained in B_1 and B_2 respectively, then $B \cap B' = \emptyset$, $B \cap B_1 = \emptyset$ and $B' \cap B_2 = \emptyset$. To see this, recall that $2B_1 \cap 2B_2 = \emptyset$, $\mathrm{diam} B \leq \frac{\mathrm{diam} B_1}{10}$ and $\mathrm{diam} B' \leq \frac{\mathrm{diam} B_2}{10}$.

We set $\mathcal{G}_j = \mathcal{F}_j \cap \mathcal{G}$ and $\mathcal{B}_j = \mathcal{F}_j \cap \mathcal{B}$. Our goal is to construct a sequence of measures (μ_n) supported on these balls. Recall that a measure μ_B can be associated to any ball B centered on E and the definition of μ_B depends on the fact that $B \in \mathcal{B}$ or $B \in \mathcal{G}$.

- $\mu_0 = a_0 \mu_{B_0}$ where $B_0 = D(0,1)$ and $a_0 > 0$ is chosen such that $\mu_0(\mathbb{C}) = \mathrm{diam} E$.
- Assume that the measures μ_1,...,μ_n have been build such that
  - (P1) $\mu_j(\mathbb{C}) = \mathrm{diam}\mathbb{C}$.
  - (P2) μ_j is supported on the disjoint (see remark 2) above) union of the balls B, $B \in \mathcal{F}_j \cup \left(\cup_{k=1}^{j-1} \mathcal{G}_k\right)$.
  - (P3) $\mu_j(B) \leq 10^{-k} R_B$ if $B \in \mathcal{F}_k$ for some $k \geq j$ (here, R_B is the radius of B).

We now construct μ_{n+1}. For this, consider $B \in \mathcal{F}_n$. As previously, there are two cases.

Case 1. $B \in \mathcal{G}$.
Set $\mu_{n+1\restriction B} = \dfrac{\mu_n(B)}{H^1(Z(B))}\mu_B$. Then, $\dfrac{\mu_n(B)}{H^1(Z(B))} \leq 100\dfrac{\mu_n(B)}{R_B} \leq 10^{-n}$ by (P2). Thus, for any ball $\tilde{B} \subset B$, $\mu_{n+1}(\tilde{B}) \leq C\text{diam}\tilde{B}$ since μ_B is Ahlfors regular. Moreover, $\mu_{n+1}(B) = \mu_n(B)$.

Case 2. $B \in \mathcal{B}$.
For any $x \in X(B)$, set $B(x) = B(x, t_B(x))$ and

$$\mu_{n+1\restriction B(x)} = \frac{\mu_n(B)}{\sum_{y\in X(B)} t_B(y)} \frac{t_B(x)}{\mu_{B(x)}(B(x))}\mu_{B(x)}.$$

Then, $\mu_{n+1}(B) = \mu_n(B)$. Note that by (25) and by induction, $\mu_{n+1}(B(x, t_B(x)) \leq 10^{-n-1}t_B(x)$.

It is not difficult to see that μ_{n+1} satisfies (P1), (P2) and (P3).

Before explaining how we get the measure μ, we recall some basic facts about weak convergence of sequences of measures (see [**65**] pages 18-19). We say that a sequence of Radon measures (ν_i) on $\mathbb{R}^n$ converges weakly to ν if $\lim_{i\to+\infty} \int \phi d\nu_i = \int \phi d\nu$ for all $\phi \in C_0(\mathbb{R}^n)$. This notion is very convenient since, if $\sup_{i\in\mathbb{N}} \nu_i(K) < +\infty$ for all compact sets $K \subset \mathbb{R}^n$, then there exists a subsequence of (ν_i) which converges weakly. Moreover, if (ν_i) converges weakly to ν, then

$$\nu(K) \geq \limsup_{i\to+\infty} \nu_i(K) \text{ for any compact } K \subset \mathbb{R}^n,$$

$$\nu(O) \leq \liminf_{i\to+\infty} \nu_i(O) \text{ for any open set } O \subset \mathbb{R}^n.$$

Since $\mu_i(\mathbb{C}) = \text{diam}E < +\infty$, there exists a weakly convergent subsequence of (μ_i). Denote by μ its weak limit. Then, set $\mu(B) = \alpha(B)R_B$. By (P3), if $B \in \mathcal{F}_k$,

$$\alpha(B) \leq 10^{-k}. \tag{33}$$

id est, $\mu(B) \leq C10^{-k}R_B$. Moreover, if $B' \subset B$ is a ball of $\mathcal{F}_{k+1}$, then $\alpha(B') \leq C\alpha(B)$. Note that (P1) implies the conclusion (i) of theorem 45 and $\mu(\mathbb{C}) \leq C\text{diam}E$.

The measure μ has linear growth

First, note that is enough to prove that $\mu(B(x, R)) \leq CR$ where $x \in \text{Supp}\mu$ and $R > 0$.
Indeed, consider $x \in \mathbb{C}$. If $B(x, R) \cap \text{Supp}\mu = \emptyset$, then $\mu(B(x, R)) = 0$. Otherwise, there exists $y \in B(x, R) \cap \text{Supp}\mu$, and therefore

$$\mu(B(x, R)) \leq \mu(B(y, 10R)) \leq CR.$$

From now on, we assume that $x \in \text{Supp}$. Then there exists a collection of balls $B_j(x)$ such that $x \in B_j(x)$ and $B_j(x) \in \mathcal{F}_j$.The collection of balls $B_j(x)$ may be finite if $x \in B$ for some $B \in \mathcal{G}$.
If $R \geq \dfrac{\text{diam}E}{10}$, then $\mu(B(x, R)) \leq \mu(\mathbb{C}) \leq C\text{diam}E \leq CR$. Assume now that $R \leq \dfrac{\text{diam}E}{10}$. Let $B_j(x)$ be the ball with the smallest radius such that $B(x, R) \subset 2B_j(x)$.

Case 1. $B_j(x) \in \mathcal{G}$. Then, since $\mu_{\upharpoonright B_j(x)}$ is Ahlfors regular with constant depending only on M, $\mu(B(x,R)) \leq CR$.

In the other cases below, $B_j(x)$ is supposed to be in $\mathcal{B}$.

Case 2. $R_{B_j(x)} \leq 10^6 R$.
Then, by (33),

$$\mu(B(x,R)) \leq \mu(2B_j(x)) \leq CR_{B_j(x)} \leq C10^6 R.$$

Case 3. $R < 10^{-6} R_{B_j(x)}$.
Set $\mathcal{K}(x,R) = \{B \in \mathcal{F}_{j+1}, B \cap B(x,R) \neq \emptyset\}$. Then, if $B \in \mathcal{K}(x,R)$, then $R_B \leq 10R$. Indeed, if not, $B(x,R) \subset 2B$ and this contradicts the choice of $B_j(x)$. Thus, any $B \in \mathcal{K}(x,R)$ is contained in $B(x,20R)$. Note that $B(x,20R) \subset 2B_j(x)$. Therefore,

$$\begin{aligned}
\mu(B(x,R)) &\leq \sum_{B \in \mathcal{K}(x,R)} \mu(B) \leq C \sum_{B \in \mathcal{K}(x,R)} R_B \text{ (by (33))} \\
&\leq C \sum_{B \in \mathcal{K}(x,R)} H^1(\Gamma_{B_j(x)} \cap B) \\
&\leq CH^1(\Gamma_{B_j(x)} \cap B(x,20R)) \\
&\leq CR \text{ (since } \Gamma_{B_j(x)} \text{ is Ahlfors regular).}
\end{aligned}$$

The measure μ has finite Menger curvature

We start with some notations and some observations.

To each ball $B = B(x, t_B(x)) \in \mathcal{B}$ corresponds a cross $G(x)$. Denote by $W_i(B)$, $i = 1,2,3,4$ the four segments in $G(x) \cap (B(x,t_B(x)) \setminus B\left(x, \frac{1}{2}t_B(x)\right)$. Let $y \in W_1(B)$. Set $y = y_1$ and denote by y_2, y_3 and y_4 the points of $W_2(B)$, $W_3(B)$ and $W_4(B)$ that you get by rotating y. We set $W(B) = \cup_{i=1}^4 W(B_i)$.

Observation 1

Let B_1, B_2 and B_3 be three balls of $\mathcal{B}$ such that $2B_i \cap 2B_j$ for $i \neq j$. Let x_1, x_2, x_3 be three points in B_1, B_2 and B_3 respectively.

Consider $z \in B_1$. Then,

$$\begin{aligned}
|x_1 - x_2| &\leq |x_1 - z| + |z - x_2| \\
&\leq 2R_{B_1} + |x_2 - z| \\
&\leq 3|x_2 - z|.
\end{aligned}$$

Therefore,

$$\frac{1}{3}|x_2 - x_1| \leq |x_2 - z| \leq 3|x_2 - x_1|$$

and, for the same reasons,

$$\frac{1}{3}|x_3 - x_1| \leq |x_3 - z| \leq 3|x_3 - x_1|.$$

By lemma 35, this implies

$$c(x_1,x_2,x_3) \leq c(z,x_2,x_3) + 10\frac{|x_1-z|}{|z-x_2||z-x_3|}.$$

Let $y_1 \in W_1(B_1)$. Then, there exists $i \in \{1,2,3,4\}$ such that $|x_1-y_i| \leq 10d(y_i, L_{x_2x_3})$. Thus,

$$\begin{aligned} c(x_1,x_2,x_3) &\leq c(y_i,x_2,x_3) + 100\frac{d(y_i,L_{x_2x_3})}{|y_i-x_2||y_i-x_3|} \\ &\leq 51c(y_i,x_2,x_3). \end{aligned}$$

Observation 2

Let B_1 and B_3 be two balls of $\mathcal{B}$ such that $2B_1 \cap 2B_3 = \emptyset$.
Let $z \in B_1$. As previously,

$$\frac{1}{3}|x_3-x_1| \leq |x_3-z| \leq 3|x_3-x_1|.$$

By lemma 35,

$$c(x_1,x_2,x_3) \leq c(z,x_2,x_3) + 10\frac{|x_1-z|}{|z-x_2||z-x_3|}.$$

Let $y_1 \in W_1(B_1)$. Then, there exists $i \in \{1,2,3,4\}$ such that $|x_1-y_i| \leq 10d(y_i, L_{x_2x_3})$. Thus, $c(x_1,x_2,x_3) \leq 51c(y_i,x_2,x_3)$.

We now prove the estimate

$$c^2(\mu) \leq C\text{diam}E.$$

First, note that, if x, y and z are three points in Suppμ, then there exists a minimal ball $B \in \mathcal{F}$ (denoted by $B(x,y,z)$) such that x, y, $z \in B$. Thus, this yields

$$c^2(\mu) \leq c^2(\mathcal{G}) + c^2(\mathcal{B}) \tag{34}$$

where $c^2(\mathcal{G}) = \sum_{B\in\mathcal{G}} c^2(B)$, $c^2(\mathcal{B}) = \sum_{B\in\mathcal{B}} c^2(B)$ and

$$c^2(B) = \int\int\int c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z)$$

where the integral is over all triples (x,y,z) such that $B(x,y,z) = B$.

Assume that $B \in \mathcal{G}_n$. Since $B \cap \text{Supp}\mu$ is contained in a Ahlfors regular curve Γ_B,

$$\int\int\int_{(B\cap\text{Supp}\mu)^3} c(x,y,z)^2 dH^1(x)dH^1(y)dH^1(z) \leq CR_B.$$

But, by construction of μ,

$$c^2(B) \leq C(\alpha(B))^3 \int\int\int_{(B\cap\text{Supp}\mu)^3} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z)$$

where $\alpha(B)$ satisfies $\mu(B) = \alpha(B)R_B$ and $\alpha(B) \leq 10^{-n}$ (see (33)). Thus, we get

$$c^2(B) \leq C(\alpha(B))^3 R_B.$$

Therefore,

$$\begin{aligned} c^2(\mathcal{G}) &= \sum_{n=0}^{+\infty} \sum_{B\in\mathcal{G}_n} c^2(B) \\ &\leq C \sum_{n=0}^{+\infty} \sum_{B\in\mathcal{G}_n} (10^{-n})^2 \alpha(B) R_B \\ &\leq C \sum_{n=0}^{+\infty} (10^{-n})^2 \sum_{B\in\mathcal{G}_n} \mu(B). \end{aligned}$$

Therefore, since the balls $B \in \mathcal{G}$ are disjoint, we get

$$c^2(\mathcal{G}) \leq C\mu(E) \leq C\mathrm{diam}E. \tag{35}$$

Assume now that $B \in \mathcal{B}_n$. Let x_1, x_2 and x_3 be three points of $B \cap \mathrm{Supp}\mu$. Denote by B_i, $i = 1, 2, 3$, the balls of $\mathcal{F}_{n+1}$ such that $B_i \subset B$ and $x_i \in B_i$, $i = 1, 2, 3$.

Case 1. the balls B_1, B_2 and B_3 are disjoint.
Then by observation 1, for any choice of points $y_i \in W(B_i)$, $i = 1, 2, 3$ there exist points $z_i \in W(B_i)$, which are obtained by rotating the y_i's, such that

$$c(x_1, x_2, x_3) \leq 200c(z_1, z_2, z_3).$$

By integrating with respect to H^1 and then with respect to μ, we get
(36)

$$c^2(B_1, B_2, B_3) \leq C \left(\Pi_{i=1}^3 \frac{\mu(B_i)}{r_{B_i}} \right) \int_{W(B_1)} \int_{W(B_2)} \int_{W(B_3)} c(z_1, z_2, z_3)^2 dH^1(z_1) dH^1(z_2) dH^1(z_3)$$

where

$$c^2(B_1, B_2, B_3) = \int_{B_1} \int_{B_2} \int_{B_3} c(x_1, x_2, x_3)^2 d\mu(x_1) d\mu(x_2) d\mu(x_3).$$

Set $c_1^2(B) = \sum_{B_1} \sum_{B_2} \sum_{B_3} c^2(B_1, B_2, B_3)$ where the sum is taken over all triples of disjoint balls B_1, B_2, $B_3 \in \mathcal{F}_{n+1}$ such that $B_i \subset B$. We have already seen that, by construction, $\dfrac{\mu(B_i)}{r_{B_i}} \leq C\alpha(B)$, $i = 1, 2, 3$. Therefore, by (36), and since Γ_B is a Ahlfors regular curve,

$$\begin{aligned} c_1^2(B) &\leq C(\alpha(B))^3 \int\int\int_{(\Gamma(B))^3} c(z_1, z_2, z_3)^2 dH^1(z_1) dH^1(z_2) dH^1(z_3) \\ &\leq C\alpha(B)^3 r_B. \end{aligned}$$

Thus,

$$\sum_{B\in\mathcal{B}} c_1^2(B) = \sum_{n=0}^{+\infty} \sum_{B\in\mathcal{B}_n} c_1^2(B) \leq C \sum_{n=0}^{+\infty} \sum_{B\in\mathcal{B}_n} (10^{-n})^2 \mu(B) \leq C\mu(E) \leq C\mathrm{diam}E. \tag{37}$$

Case 2. $B_1 = B_2$ and $B_3 \neq B_1$.
Then by observation 1 and observation 2, for any choice of points $y_i \in W(B_i)$, $i = 1, 2, 3$, there exist points $z_i \in W(B_i)$, which are obtained by rotating the y_i's, such that

$$c(x_1, x_2, x_3) \leq 200c(z_1, z_2, z_3).$$

The rest of the proof is similar to those of the case 1.
By integrating with respect to H^1 and then with respect to μ and if we set $c^2(B_1, B_2, B_3) = \int_{B_1}\int_{B_2}\int_{B_3} c(x_1, x_2, x_3)^2 d\mu(x_1)d\mu(x_2)d\mu(x_3)$, we get
(38)

$$c^2(B_1, B_2, B_3) \leq C\left(\Pi_{i=1}^3 \frac{\mu(B_i)}{r_{B_i}}\right)\int_{W(B_1)}\int_{W(B_2)}\int_{W(B_3)} c(z_1, z_2, z_3)^2 dH^1(z_1)dH^1(z_2)dH^1(z_3).$$

Set $c_2^2(B) = \sum_{B_1}\sum_{B_2}\sum_{B_3} c^2(B_1, B_2, B_3)$ where the sum is taken over all triples of balls of B satisfying $B_1 = B_2$ and $B_3 \neq B_1$. Then, by (38),

$$\begin{aligned} c_2^2(B) &\leq C(\alpha(B))^3 \int\int\int_{(\Gamma_B)^3} c(z_1, z_2, z_3)^2 dH^1(z_1)dH^1(z_2)dH^1(z_3) \\ &\leq C\alpha(B)^3 r_B. \end{aligned}$$

Thus,

$$\sum_{B\in\mathcal{B}} c_2^2(B) = \sum_{n=0}^{+\infty}\sum_{B\in\mathcal{B}_n} c_2^2(B) \leq C\sum_{n=0}^{+\infty}\sum_{B\in\mathcal{B}_n}(10^{-n})^2\mu(B) \leq C\mu(E) \leq C\text{diam}E. \tag{39}$$

Finally, (34), (35), (37) and (39) give $c^2(\mu) \leq C\text{diam}E$ and the proof of theorem 45 is complete. □

CHAPTER 4

The Cauchy singular integral operator on Ahlfors regular sets

The main goal of this chapter is to show that the L^2-boundedness of the Cauchy operator on an Ahlfors-regular set in $\mathbb{C}$ is closely related to the rectifiability properties of the set. For the convenience of the reader, we also include an overview of the theory of Calderón-Zygmund operators and a proof (due to M. Melnikov and J. Verdera) of the L^2-boundedness of the Cauchy operator on Lipschitz graphs. However, for more details about the material described in this chapter, the reader is urged to consult [**17**], [**24**], [**32**], [**73**], [**95**] and [**96**] for instance. We should mention that the definitions of standard kernels, singular integral operators, Calderón-Zygmund operators we will give may differ from those given in these books.

1. The Hilbert transform

We start with an elementary, but very instructive example of singular integral operator, namely the Hilbert transform in $\mathbb{R}$. Formally, the Hilbert transform of a function f is given by

$$Hf(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{x-y} d\mathcal{L}^1(y).$$

The first problem is the convergence of the integral. For this, assume that $f \in C_0^1(\mathbb{R})$ (that is f is a C^1 function with compact support). The Hilbert transform of f (in the principal value sense) is

$$\begin{aligned} Hf(x) &= \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} d\mathcal{L}^1(y) \\ &=: \frac{1}{\pi} p.v. \int \frac{f(y)}{x-y} d\mathcal{L}(y). \end{aligned}$$

(p.v. means principal value).

We claim that the limit exists. Indeed, since $\displaystyle\int_{|x-y|>\varepsilon} \frac{1}{x-y} d\mathcal{L}^1(y) = 0$, we get (if we assume that supp$f \subset [-1,1]$)

$$\int_{\varepsilon<|x-y|\leq 1} \frac{f(y)}{x-y} d\mathcal{L}^1(y) = \int_{\varepsilon<|x-y|\leq 1} \frac{f(y)-f(x)}{x-y} d\mathcal{L}^1(y).$$

Now write

$$\int_{\varepsilon<|x-y|\leq 1} \frac{f(y)-f(x)}{x-y} d\mathcal{L}^1(y) = \int_{|x-y|\leq 1} \frac{f(y)-f(x)}{x-y} d\mathcal{L}^1(y) - \int_{|x-y|\leq \varepsilon} \frac{f(y)-f(x)}{x-y} d\mathcal{L}^1(y).$$

Since f is a C^1 function, the first integral of the right hand side is finite and the second one tends to 0 with ε. This completes the proof of the claim.
Note that, if we replace the kernel $\dfrac{1}{x-y}$ by $\dfrac{1}{|x-y|}$, the principal value limit would

H. Pajot: LNM 1799, pp. 55–65, 2002.

fail to exist. Thus, cancellation properties of the kernel play an important role.

Now it is possible to extend the Hilbert transform to $L^2(\mathbb{R})$ by means of the following formula

$$(*)\widehat{Hf}(\xi) = -i\widehat{f}(\xi)\frac{\xi}{|\xi|}$$

whenever $\xi \in \mathbb{R}^*$. Recall that $\widehat{f}(\xi) = \int e^{-2i\pi x.\xi} f(x)d\mathcal{L}^1(x)$ and note that (*) is true for functions in $C_0^1(\mathbb{R})$. From Plancherel's theorem and the density of functions of $C_0^1(\mathbb{R})$ in $L^2(\mathbb{R})$, it follows that H has an unique extension to a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

2. Singular integral operators

The study of singular integral operators in higher dimensions was initiated by A. Zygmund and his student A. P. Calderón. Natural generalizations of the Hilbert transform in $\mathbb{R}^n$ are Riesz transforms:

$$R_j f(x) = c_n \int_{\mathbb{R}^n} f(y)\frac{x_j - y_j}{|x-y|^{n+1}} d\mathcal{L}^n(y)$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}$ and Γ is the classical Gamma function.
In their fundamental paper [**13**], A. Zygmund and A. P. Calderón consider "kernels" $\Omega : \mathbb{R}^n \to \mathbb{R}$ such that
(i) Ω is homogeneous of degree 0, that is $\Omega(\lambda x) = \Omega(x)$ for any $x \in \mathbb{R}^n$, any $\lambda > 0$.
(ii) $\int_{S^{n-1}} \Omega(y)dy = 0$ where S^{n-1} is the standard unit sphere of $\mathbb{R}^n$.
(iii) $|\Omega(y) - \Omega(y')| \leq \omega(|y - y'|)$ whenever y, y' are in S^{n-1} and where ω is a non decreasing function such that $\int_0^1 \frac{\omega(t)}{t} d\mathcal{L}^1(t) < \infty$. A basic example of such a function ω is $\omega(t) = t^\alpha$ where $\alpha > 0$.
Then, they define the family of operators

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(y-x)d\mathcal{L}^n(y)$$

and they prove that, if $p > 1$, $||T_\varepsilon f||_{L^p(\mathbb{R}^n)} \leq A_p||f||_{L^p(\mathbb{R}^n)}$ where $A_p > 0$ does not depend on ε. From this, they get that the sequence (T_ε) tends (with respect to the L^p topology) to an operator T and that this operator T is bounded on $L^p(\mathbb{R}^n)$.
The study of the Hilbert transform involves a lot of methods from complex analysis. For instance, the proof of M. Riesz of the L^p boundedness of the Hilbert transform is based on contour integral techniques. By contrast, A. P. Calderón and A. Zygmund introduce real-variable methods, for instance what is now called the Calderón-Zygmund lemma and the Calderon-Zygmund decomposition (see [**95**] for these notions and [**97**] for an history of these developments).
We now describe what Y. Meyer and R. Coifman call "les nouveaux opérateurs de Calderón-Zygmund" (see [**73**]). These operators differ from those considered by A. Zygmund and A. P. Calderón by the fact that they are not convolution operators.

Given an integer $0 < d \leq n$, a standard kernel with homogeneity d is a function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y), x \neq y\} \to \mathbb{C}$ such that for some constants $\delta > 0$ and $M > 0$ and for $x \neq y$,

(i) $|K(x,y)| \leq \dfrac{M}{|x-y|^d}$;

(ii) $|K(x,y) - K(x',y)| \leq M\dfrac{|x'-x|^\delta}{|x-y|^{d+\delta}}$ if $|x'-x| < \dfrac{1}{2}|x-y|$,

(iii) $|K(x,y) - K(x,y')| \leq M\dfrac{|y'-y|^\delta}{|x-y|^{d+\delta}}$ if $|y'-y| < \dfrac{1}{2}|x-y|$.

We say that K is antisymmetric if $K(x,y) = -K(y,x)$ whenever $x \neq y$. A basic example of antisymmetric standard kernel is $\frac{1}{x-y}$ in $\mathbb{R}$, that is the kernel associated to the Hilbert transform. Note that, if K is antisymmetric, (iii) follows from (ii). For simplicity, we will only consider antisymmetric kernels.

Let μ be a positive Radon measure on $\mathbb{R}^n$ and let K be an antisymmetric standard kernel. The singular integral operator T associated to K and applied to μ is defined formally by

$$Tf(x) = \int f(y)K(x,y)d\mu(y).$$

Remark. By analogy with the Hilbert transform, the integer d in the definition of the standard kernel K should be seen as the "dimension" of the measure μ. In other words, a natural setting (see e.g. [**28**], [**29**] or [**68**]) to study a singular integral operator associated to a kernel K and applied to a measure μ is when K is d-homogeneous and μ is Ahlfors regular with dimension d, that is there exists $C_0 > 0$ such that

$$C_0^{-1}R^d \leq \mu(B(x,R)) \leq C_0R^d$$

whenever $x \in \text{Supp}\mu$, $R \in (0, \text{diam}(\text{Supp})\mu)$.

Like for the Hilbert transform, we define the truncated singular integral operator T_ε by

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} f(y)K(x,y)d\mu(y).$$

We say that T is bounded on $L^p(\mu)$ if the operators T_ε are bounded on $L^p(\mu)$ uniformly on $\varepsilon > 0$.

Digression: The previous definition is very convenient because it allows us to speak about the L^p boundedness to T without giving a precise sense of T ! In most cases, it is impossible to define the singular integral operator in the principal value sense like for the Hilbert transform. Another way consists in using the theory of distributions. Following R. Coifman and Y. Meyer [**73**], a singular integral operator T associated to an antisymmetric standard kernel K is a bounded linear operator from $C_0^\infty(\mathbb{R}^n)$ to its dual given by

$$< Tf, g >= \frac{1}{2}\int\int K(x,y)[f(y)g(x) - f(x)g(y)]d\mu(x)d\mu(y).$$

Note that the singularity of the kernel K for $x = y$ is killed by the estimate $|f(y)g(x) - g(y)f(x)| \leq C|x-y|$ (where C depends on f and g).
A Calderón-Zygmund operator is a singular integral operator which can be extended to a bounded operator of $L^2(\mu)$. We denote by $||T||_{CZ}$ its norm as such operator. More generally, we say that T is bounded on $L^p(\mu)$ if T extends to a bounded operator on $L^p(\mu)$. Anyway, if μ is not atomic, this definition of L^p boundedness of T is equivalent to the uniform L^p boundedness of the truncated operators T_ε (see e.g. [**24**]).

For more details about the relationship between the existence of singular integral operators and the geometry of measures or sets, see the recent survey [**68**].

3. The Hardy-Littlewood maximal operator

Let μ be a positive Radon measure in $\mathbb{R}^n$. The Hardy-Littlewood maximal operator M_μ related to μ is defined by

$$M_\mu f(x) = \sup_{R>0} \frac{1}{\mu(B(x,R))} \int_{B(x,R)} |f(y)| d\mu(y).$$

The Hardy-Littlewood maximal operator is not of course a singular integral operator, but is a very useful tool in the Calderón-Zygmund theory. We will illustrate this by giving a version of Cotlar inequality at the end of this section.

THEOREM 46. *Let μ be a doubling positive Radon measure in $\mathbb{R}^n$. Then, M_μ is bounded on $L^p(\mu)$ for all $1 < p < \infty$ and is of weak type $(1,1)$.*

Recall that an operator T is of weak type $(1,1)$ (with respect to a positive Radon measure μ) if for all $f \in L^1(\mu)$ and all $\lambda > 0$,

$$\mu(\{x \in \mathbb{R}^n; |Tf(x)| > \lambda\}) \le C \frac{||f||_{L^1(\mu)}}{\lambda}.$$

PROOF. The proof is very instructive, since it shows why the notion of doubling measure is very convenient (but not necessary) to do analysis.
We first recall a basic covering theorem (see [**54**] page 2).

THEOREM 47. *Let $\mathcal{F}$ be a family of balls in a metric space X such that $\sup_{B\in\mathcal{F}} diam B < +\infty$. Then, there exists a subfamily $\mathcal{G}$ of $\mathcal{F}$ such that*
- The balls of $\mathcal{G}$ are pairwise disjoint;
- $\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B$, where $5B$ is the ball with the same center as B but whose diameter is $5 diam B$.

Fix $t > 0$. For each $x \in \{y \in \mathbb{R}^n; M_\mu f(y) > t\}$, there exists $R = R(x)$ such that $\int_{B(x,R)} |f(y)| d\mu(y) > t\mu(B(x,R))$. By the previous covering theorem, there exists a countable family of pairwise disjoint balls $B(x,R(x))$, $x \in \mathcal{G}$, such that $\{y \in \mathbb{R}^n; M_\mu f(y) > t\} \subset \bigcup_{x\in\mathcal{G}} B(x, 5R(x))$.
From this, we get

$$\begin{aligned} \mu(\{y \in \mathbb{R}^n; M_\mu f(y) > t\}) &\le \sum_{x\in\mathcal{G}} \mu(B(x,5R(x)) \\ &\le C \sum_{x\in\mathcal{G}} \mu(B(x,R(x)) \text{ (since } \mu \text{ is doubling)} \\ &\le \frac{C}{t} \sum_{x\in\mathcal{G}} \int_B |f(y)| d\mu(y) \text{ (by definition of } R(x)) \\ &\le \frac{C}{t} \int |f(y)| d\mu(y) \text{ (since the balls } B(x,R(x)) \text{ are disjoint)} \end{aligned}$$

The L^p boundedness of M_μ follows from the weak $(1,1)$ boundedness and the following observations:

$$\{M_\mu f > t\} \subset \{M_\mu(f.\chi_E) > \frac{t}{2}\}, \text{ where } E = \{f > \frac{t}{2}\}$$

$$\int |M_\mu f|^p d\mu = p \int_0^{+\infty} t^{p-1} \mu(M_\mu f > t) dt.$$

The last equality is usually called "Layer cake representation" (see [**57**]).
Note that this proof works also in a metric space (X, d) which carries a doubling measure μ (see [**54**] for details). □

Let T be a Calderón-Zygmund operator in the sense of the digression of the previous section (applied to $\mu = \mathcal{L}^n$). Define $T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$. Then, for every $x \in \mathbb{R}^n$,

$$T^* f(x) \leq C(M(Tf)(x) + ||T||_{CZ} Mf(x))$$

where M is the maximal operator associated to $\mu = \mathcal{L}^n$. In particular, this implies that the operator T^* is bounded on $L^2(\mathbb{R}^n)$.
See [**52**] for more details and other versions of the Cotlar inequality.

4. The Calderón-Zygmund theory

We say that the singular integral operator T is of weak type $(1, 1)$ (with respect to a positive Radon measure μ) if there exists $C > 0$ such that for all $f \in L^1(\mu)$, all $\lambda > 0$, all $\varepsilon > 0$,

$$\mu(\{x \in \mathbb{R}^n; |T_\varepsilon f(x)| > \lambda\}) \leq C \frac{||f||_{L^1(\mu)}}{\lambda}.$$

A very important fact in the theory of singular integral operators as developed by A. Calderón and A. Zygmund is the following result.

THEOREM 48 (Calderón-Zygmund). *If μ is a doubling measure and if T is a singular integral operator on $\mathbb{R}^n$ which is bounded on $L^2(\mu)$, then T is bounded on $L^p(\mu)$ for $p \in (1, +\infty)$ and is of weak type* $(1, 1)$.

Therefore, one important problem concerning singular integral operators is to study their L^2 boundedness. We will see in the next section two powerful criteria due to G. David, J. L. Journé and S. Semmes.

Before, we should mention that Calderón-Zygmund theory has a lot of applications to partial differential equations, potential theory and complex analysis (see for instance the talks given at the ICM by C. Fefferman [**39**], A. P. Calderón [**12**], Y. Meyer [**74**] and G. David [**23**]) and has also two more surprising applications:
- The resolution of the Beltrami equation for quasi-conformal mappings [**3**].
- The Atiyah-Singer index theorem [**5**].

5. The $T1$ and the Tb theorems

Let μ be a doubling positive Radon measure on $\mathbb{R}^n$. We say that $f \in BMO(\mu)$ if

$$||f||_{BMO(\mu)} = \sup_{B \subset \mathbb{C} \text{ balls}} \frac{1}{\mu(B)} \int_B |f - f_B| d\mu < +\infty$$

where $f_B = \dfrac{1}{\mu(B)} \displaystyle\int_B f d\mu$. Then, $||f||_{BMO(\mu)}$ is a semi-norm on $BMO(\mu)$, but is a norm on $BMO(\mu)/\{constantfunctions\}$ (that we call also $BMO(\mu)$). Note that $L^\infty(\mu) \subset BMO(\mu)$.

THEOREM 49 ($T1$ Theorem of David-Journé). *Let μ be a doubling positive Radon measure on $\mathbb{R}^n$ with polynomial growth and let T be a singular integral operator (associated to an antisymmetric standard kernel). Then, $T1$ belongs to $BMO(\mu)$ if and only if T is bounded on $L^2(\mu)$.*

Comments.
1) Recall that μ has polynomial growth if there exist $d > 0$ and $C > 0$ such that $\mu(B(x,r)) \leq Cr^d$ whenever $x \in \mathbb{R}^n$, $r > 0$. G. David [**22**] proved that the polynomial growth of μ is necessary for the L^2 boundedness of singular integral operators associated to kernels which satisfy $|K(x,y)| \geq C|x-y|^{-d}$, like for instance the Hilbert transform. Thus, this hypothesis is quite natural.
2) What is the meaning of $T(1)$? One way consists to use the theory of distributions. Given a bounded continuous function b (for instance, the constant function equal to 1) Tb is a continuous linear form defined on the space of functions $g \in C_0^\infty(\mathbb{R}^n)$ with $\int g = 0$. More precisely, given $g \in C_0^\infty(\mathbb{R}^n)$, write $b = b_1 + b_2$ where $b_1 \in C_0^\infty(\mathbb{R}^n)$ and b_2 is zero in a neighborhood of Supp$g$. Then, Tb is the distribution given by

$$< Tb, g >=< Tb_1, g > + \int\int [K(x,y) - K(x_0,y)]b_2(y)g(x)d\mu(x)d\mu(y)$$

where x_0 is a point of Supp$g$. Note that this definition does not depend on x_0, the decomposition of b and coincides with the definition of Tb when $b \in \mathcal{D}(\mathbb{R}^n)$.
On the other hand (see [**52**]), $T1 \in BMO(\mu)$ if and only if for all balls B in $\mathbb{R}^n$,

$$\int_B |T(\chi_B)|d\mu \leq C\mu(B).$$

3) If the kernel of T is not antisymmetric, we should require also that $T^t 1$ (where T^t is the transpose of T) belongs to $BMO(\mu)$ and that T is weakly bounded (see [**24**] for instance).

A more flexible criterion for the L^2 boundedness of singular integral operators is the $T(b)$ theorem of David-Journé-Semmes.
Here, $b : \mathbb{R}^n \to \mathbb{C}$ is a bounded function such that there exists $C > 0$ and $\delta > 0$ such that for every $x \in \mathbb{R}^d$, every $r > 0$, there exists a cube Q with

$$d(x,Q) \leq Cr, \ \frac{r}{C} \leq \text{diam}Q \leq r, \ \text{and} \ \left|\frac{1}{|Q|}\int_Q b(y)d\mu(y)\right| \geq \delta.$$

Such a function is called "para-accretive".

Remark. We will see later that the L^2 boundedness of the Cauchy operator C_Γ on Lipschitz graphs can be easily proved by applying the Tb theorem with a natural choice of b. On the other hand, it is hard to check that the hypothesis of the $T1$ theorem for C_Γ are satisfied (except if you know what is the Menger curvature ! see section 6).

THEOREM 50 (Tb theorem of David-Journé-Semmes). *Let μ be a doubling positive Radon measure on $\mathbb{R}^n$ with polynomial growth and let T be a singular integral operator (associated to an antisymmetric standard kernel). If there exists a para-accretive function b such that Tb belongs to $BMO(\mu)$, then T is bounded on $L^2(\mu)$.*

For a nice proof of the $T1$ and the Tb theorems, see [**24**].

We conclude this section with a local version of the Tb theorem due to M. Christ [**18**]. Before stating his result, we need a definition. Let $E \subset \mathbb{R}^n$ be a d-dimensional Ahlfors-regular set. We have seen that there exists a family $\Delta(E)$ of "dyadic squares" associated to E. A system of functions $\{b_Q, Q \in \Delta(E)\}$ is pseudo-accretive if there exists $C > 0$, $\varepsilon > 0$ such that for every $Q \in \Delta(E)$, $||b_Q||_\infty \le C$ and $|\int_Q b_Q dH^d| \ge \varepsilon H^d(Q)$.

THEOREM 51. *Let $E \subset \mathbb{R}^n$ be an Ahlfors-regular set with dimension d and let T be a singular integral operator associated to an antisymmetric kernel and defined with respect to $H^d_{\lceil E}$. Suppose that there exists a pseudo-accretive system $(b_Q)_{Q\in\Delta(E)}$ such that $||Tb_Q||_\infty \le C$ for any $Q \in \Delta(E)$. Then, T is bounded on $L^2(H^d_{\lceil E})$.*

This statement can be extended to spaces of homogeneous type (see [**18**]). If you compare with the original Tb theorem, the main point is that you have to construct a function b_Q associated to each Q ("what you win"), but you have to check that Tb_Q belongs to L^∞ instead of BMO ("what you lose").

Remark. If b is special para-accretive with respect to $\Delta(E)$, that is if there exists $\delta > 0$ such that $\left|\frac{1}{H^1(Q)}\int_Q b(y)dH^1(y)\right| \ge \delta$ for any $Q \in \Delta(E)$, then $(b.\chi_Q)$ is a para-accretive system.

6. L^2 boundedness of the Cauchy singular operator on Lipschitz graphs

Let Γ be a Lipschitz graph in $\mathbb{C}$, that is Γ is a subset of $\mathbb{C}$ of the form

$$\Gamma = \{x + iA(x), x \in \mathbb{R}\} \text{ where } A : \mathbb{R} \to \mathbb{R} \text{ is Lipschitz.}$$

We set $\gamma(x) = x + iA(x)$ and let ds be the arc length measure on Γ.
Let $f \in L^2(\Gamma, ds)$. The Cauchy integral of f along Γ is given formally by

$$\mathcal{C}_\Gamma f(z) = \frac{1}{2i\pi}\int_\Gamma \frac{f(\xi)}{z-\xi} ds(\xi).$$

To give a more precise sense of the Cauchy operator on Γ, define a continuous linear operator on the space of Lipschitz functions on Γ by

$$< \mathcal{C}_\Gamma f, g >= \frac{1}{2}\int_\Gamma\int_\Gamma \frac{1}{x-y}[f(x)g(y) - f(y)g(x)]ds(x)ds(y).$$

Note that this integral converges since Γ is Ahlfors-regular, and f, g are Lipschitz functions on Γ. By using the density of Lipschitz functions in $L^2(\Gamma, ds)$, we can easily extend this operator as an operator on $L^2(\Gamma, ds)$.
Set now $F(x) = f(\gamma(x))(1 + A'(x))$. Then, $||f||_{L^2(\Gamma,ds)}$ is comparable to $||F||_{L^2(\mathbb{R})}$ and, for $z = \gamma(x)$, we have formally

$$\mathcal{C}_\Gamma f(z) = \frac{1}{2i\pi}\int_\mathbb{R} \frac{F(y)}{x - y + i(A(x) - A(y))} d\mathcal{L}^1(y).$$

Define

$$\mathcal{C}^\varepsilon_\Gamma g(x) = \int_{|y-x|>\varepsilon} \frac{g(y)}{\gamma(x)-\gamma(y)} d\mathcal{L}^1(y).$$

Then (see [**17**]), $\mathcal{C}_\Gamma$ is bounded on $L^2(\Gamma, ds)$ if and only if $\mathcal{C}^\varepsilon_\Gamma$ is bounded on $L^2(\mathbb{R})$ uniformly in ε. In this case, we say that the Cauchy operator is bounded on $L^2(\Gamma)$.

The L^2 boundedness of the Cauchy operator on Lipschitz graphs has been conjectured by A. Zygmund and A. P. Calderón in the 60's. In 1977, A. P. Calderón [**11**] solved the problem when the Lipschitz constant of A is small enough. The general case was solved by R. Coifman, A. MacIntosh and Y. Meyer [**69**]. There are now a lot of different proofs of this famous theorem (see, for instance, [**18**], [**20**], [**22**], [**49**], [**76**]. The proof that we present is due to M. Melnikov and J. Verdera [**72**]. First, recall that Γ satisfies the local curvature condition (see theorem 30).

Note that, by the $T1$ theorem, it is enough to prove that, for any interval $I \subset \mathbb{R}$,

$$\int_I |\mathcal{C}_\Gamma^\varepsilon(\chi_I(x))|^2 d\mathcal{L}^1(x) \leq C|I|$$

where $C > 0$ does not depend on I and where $|I|$ denotes the length of I.

But,

$$\begin{aligned}\int_I |\mathcal{C}_\Gamma^\varepsilon(\chi_I(x))|^2 d\mathcal{L}^1(x) &= \int_I \mathcal{C}_\Gamma^\varepsilon(\chi_I(x))\overline{\mathcal{C}_\Gamma^\varepsilon(\chi_I(x))} d\mathcal{L}^1(x) \\ &= \int_I \int_{I_\varepsilon} \int_{I_\varepsilon} \frac{d\mathcal{L}^1(x) d\mathcal{L}^1(y) d\mathcal{L}^1(z)}{\overline{(\gamma(y)-\gamma(x))\overline{(\gamma(z)-\gamma(x))}}}\end{aligned}$$

where $I_\varepsilon(x) = \{t \in I; |t-x| > \varepsilon\}$.

We now symmetrize the domain by considering

$$S_\varepsilon = \{(x,y,z) \in I^3; |x-y| > \varepsilon, |x-z| > \varepsilon, |y-z| > \varepsilon\}.$$

Thus, we get

$$\int_I |\mathcal{C}_\Gamma^\varepsilon(\chi_I(x))|^2 d\mathcal{L}^1(x) = \int\int\int_{S_\varepsilon} \frac{d\mathcal{L}^1(x) d\mathcal{L}^1(y) d\mathcal{L}^1(x)}{\overline{(\gamma(y)-\gamma(x))\overline{(\gamma(z)-\gamma(x))}}} + O(|I|).$$

See the proof of (40) below for more details about this estimate.

By permutating the position of the three variables in the integral of the right hand side (recall (16)) and by (13), we get

$$\int_I |\mathcal{C}_\Gamma^\varepsilon(\chi_I(x))|^2 d\mathcal{L}^1(x) \leq C \int\int\int_{S\varepsilon} c(\gamma(x),\gamma(y),\gamma(z))^2 d\mathcal{L}^1(x) d\mathcal{L}^1(y) d\mathcal{L}^1(z) + O(|I|).$$

Theorem 30 yields

$$\int_I |\mathcal{C}_\Gamma^\varepsilon(\chi_I(x))^2| d\mathcal{L}^1(x) \leq C|I|,$$

and we conclude by using the $T1$ theorem.

Remarks.
1) The L^2-boundedness of the Cauchy operator on Lipschitz graphs can be easily proved by using the Tb theorem with $b(x) = 1 + iA'(x)$. To see this, note that b is defined almost everywhere on $\mathbb{R}$ (by the Rademacher theorem), b is para-accretive since $Re(b) \geq 1$ (such a function is usually called accretive), and

$$\int \frac{b(y)}{\gamma(x)-\gamma(y)} d\mathcal{L}^1(y) = 0.$$

2) A classical result of I. Privalov [**92**] (see also [**33**] page 244) says that, except a set of zero measure, the Cauchy integral $f(z) = \int_\Gamma \frac{\phi(\xi)}{\xi - z} d\xi$ (where Γ is a rectifiable Jordan curve) has a non tangential limit at z_0 if and only if the Cauchy singular operator $\mathcal{C}_\Gamma$ exists at z_0 in the principal value sense. If Γ is a Lipschitz graph (or equivalently a Lipschitz curve), the L^2-boundedness of the Cauchy operator on Γ implies that $p.v. \int_\Gamma \frac{\phi(\xi)}{\xi - z} d\xi$ exists almost everywhere for $\phi \in L^2(\Gamma, ds)$ (see the nice discussion in [**106**]).

3) The L^2 boundedness of the Cauchy operator on Lipschitz graphs has a lot of applications to partial differential equations on non smooth domains (see [**55**]).

4) If Γ is a line, the L^2 boundedness of the Cauchy operator on Γ follows from Plancherel formula (see the section about the Hilbert transform). If Γ is a (general) Lipschitz graph, then Γ looks like a straight line at most places at most scales. This can be stated in terms of beta numbers (see theorem 18). The proof of the L^2 boundedness of the Cauchy operator on Lipschitz graphs given by P. Jones in [**49**] uses these two observations.

7. Cauchy singular operator and rectifiability

Let $E \subset \mathbb{C}$ be an Ahlfors-regular set with dimension 1. As previously, we define the truncated Cauchy operator by

$$\mathcal{C}_E^\varepsilon f(x) = \int_{|y-x|>\varepsilon} \frac{f(y)}{y - x + i(y-x)} d\mu(y),$$

where μ is the restriction of the one dimensional Hausdorff measure to E.
In this section, we discuss the following problem: under which conditions is the Cauchy operator $\mathcal{C}_E$ bounded on $L^2(\mu)$? We will see that this question has a nice answer in terms of rectifiability.

Recall that an Ahlfors-regular set E is said to be uniformly rectifiable if E is contained in an Ahlfors-regular curve. In the previous chapter (see theorem 39), we proved that E is uniformly rectifiable if and only if E satisfies the local curvature condition:

$$\int_B \int_B \int_B c(x,y,z)^2 d\mu(x) d\mu(y) d\mu(z) \leq C \mathrm{diam} B$$

for any ball B such that $E \cap B \neq \emptyset$.

THEOREM 52 ([**69**]). *Let $E \subset \mathbb{C}$ be a Ahlfors-regular set (with dimension 1). The following assertions are equivalent*
(i) E is uniformly rectifiable.
(ii) $\mathcal{C}_E$ is bounded on $L^2(\mu)$.

PROOF. The proof of $(i) \to (ii)$ is a straightforward modification of the proof of M. Melnikov and J. Verdera in the case of Lipschitz graphs, by noting that the properties of Lipschitz graphs that they use are the Ahlfors-regularity and the local curvature condition.

For the convenience of the reader, we give some details of the proof.
Since (E, μ) is a space of homogeneous type, we can use the $T1$ theorem. Therefore, it is enough to show that

$$\int_B |\mathcal{C}_E^\varepsilon \chi_B|^2 d\mu \leq C \text{diam} B$$

whenever B is a ball in $\mathbb{C}$.
But,

$$\begin{aligned} \int_B |\mathcal{C}_E^\varepsilon \chi_B(x)|^2 d\mu(x) &= \int_B \mathcal{C}_E^\varepsilon \chi_B(x) \overline{\mathcal{C}_E^\varepsilon \chi_B(x)} d\mu(x) \\ &= \int_B \int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} \frac{1}{(y-x)\overline{(z-x)}} d\mu(x) d\mu(y) d\mu(z), \end{aligned}$$

where $B_\varepsilon(x) = \{u \in B; |u-x| > \varepsilon\}$.
Consider $S_\varepsilon = \{(x,y,z) \in B^3; |x-y| > \varepsilon, |x-z| > \varepsilon, |y-z| > \varepsilon\}$. Then, by using the Ahlfors regularity of μ (see the proof of (40) below), we get

$$\begin{aligned} \int_B |\mathcal{C}_E^\varepsilon \chi_B(x)|^2 d\mu(x) &= \int\int\int_{S_\varepsilon} \frac{1}{(y-x)\overline{(z-x)}} d\mu(x) d\mu(y) d\mu(z) + O(\text{diam} B) \\ &\leq C \int\int\int_{S_\varepsilon} c(x,y,z)^2 d\mu(x) d\mu(y) d\mu(z) + O(\text{diam} B) \text{ by (13)} \\ &\leq C \text{diam} B \text{ by theorem 33} \end{aligned}$$

To prove the converse, fix a ball B in $\mathbb{C}$ and note that by the L^2 boundedness of the Cauchy operator applied to the constant function $f = 1$, we get

$$I =: \int_{(E\cap B)} \left| \int_{(E\cap B)\setminus B(z,\varepsilon)} \frac{1}{\xi - z} dH^1(\xi) \right|^2 dH^1(z) \leq C \text{diam} B.$$

Write $I = I_1 + I_2$ where

$$I_1 = \int\int\int_{S_\varepsilon} \frac{1}{(\zeta - z)\overline{(\xi - z)}} dH^1(z) dH^1(\zeta) dH^1(\xi),$$

$$I_2 = \int\int\int_{T_\varepsilon} \frac{1}{(\zeta - z)\overline{(\xi - z)}} dH^1(z) dH^1(\zeta) dH^1(\xi),$$

$$S_\varepsilon = \{(z,\zeta,\xi) \in (E\cap B)^3; |z-\zeta| > \varepsilon, |z-\xi| > \varepsilon, |\zeta - \xi| > \varepsilon\},$$
$$T_\varepsilon = \{(z,\zeta,\xi) \in (E\cap B)^3; |z-\zeta| > \varepsilon, |z-\xi| > \varepsilon, |\zeta - \xi| \leq \varepsilon\}.$$

Moreover, by the "magic" formula (13), $I_1 = \dfrac{1}{6} \displaystyle\int\int\int_{S_\varepsilon} c(x,\zeta,\xi)^2 dH^1(z) dH^1(\zeta) dH^1(\xi)$ and therefore, $I_1 \longrightarrow \frac{1}{6} c^2(\mu)$ when $\varepsilon \longrightarrow 0$. Furthermore, we claim that

$$I_2 = O(\text{diam} B). \tag{40}$$

Indeed, write

$$\begin{aligned} I_2 &= \int\int\int_{U_\varepsilon} \frac{1}{(\zeta - z)\overline{(\xi - z)}} dH^1(z) dH^1(\zeta) dH^1(\xi) \\ &+ \int\int\int_{V_\varepsilon} \frac{1}{(\zeta - z)\overline{(\xi - z)}} dH^1(z) dH^1(\zeta) dH^1(\xi) \end{aligned}$$

where $U_\varepsilon = \{(z,\zeta,\xi) \in (E\cap B)^3; |z-\zeta| > \varepsilon, \varepsilon < |z-\xi| \leq 2\varepsilon, |\zeta-\xi| \leq \varepsilon\}$ and $V_\varepsilon = \{(z,\zeta,\xi) \in (E\cap B)^3; |z-\zeta| > \varepsilon, |z-\xi| > 2\varepsilon, |\zeta-\xi| \leq \varepsilon\}$.
Note that, if $(z,\zeta,\xi) \in V_\varepsilon$, $|z-\xi| \leq 2|z-\zeta|$.
By using the regularity of E, we easily get

$$\begin{aligned}
\left| \int\int\int_{U_\varepsilon} \frac{1}{(\zeta-z)\overline{(\xi-z)}} dH^1(z)dH^1(\zeta)dH^1(\xi) \right| &\leq C\mathrm{diam}B, \\
\left| \int\int\int_{V_\varepsilon} \frac{1}{(\zeta-z)\overline{(\xi-z)}} dH^1(z)dH^1(\zeta)dH^1(\xi) \right| &\leq \int\int\int_{V_\varepsilon} \frac{2}{|z-\xi|^2} dH^1(z)dH^1(\zeta)dH^1(\xi) \\
&\leq C\mathrm{diam}B.
\end{aligned}$$

This yields

$$\int\int\int_{E\cap B} c(z,\zeta,\xi)^2 dH^1(z)dH^1(\zeta)dH^1(\xi) \leq C\mathrm{diam}B.$$

Thus, E satisfies the local curvature condition and hence E is uniformly rectifiable (by theorem 39). □

Open problem. Let d be an integer with $d > 1$ and let $E \subset \mathbb{R}^n$ be a Ahlfors regular set with dimension d.
Is it true that E is uniformly rectifiable if and only if the singular integral operator associated to the kernel $|x|^{d-1}x$ is bounded on $L^2(E, dH^d_{\lceil E})$? See [**28**], [**29**] and [**68**] for the definitions and related results. The main problem for extending the method described above is that it seems impossible to find a Menger type curvature associated to this kernel (see the discussion in [**37**]).

CHAPTER 5

Analytic capacity and the Painlevé problem

From our point of view, the history before 1950 of the notion of removability in complex analysis can be summed up as follows. In 1851, B. Riemann stated his famous theorem on removable singularities. Later, in 1880, P. Painlevé was interested in a more general problem and asked for a necessary and sufficient condition for a compact set $E \subset \mathbb{C}$ to be removable for bounded analytic functions. In 1947, L. Ahlfors introduced the notion of analytic capacity and proved that removable sets for bounded analytic functions are exactly sets with vanishing analytic capacity. In this chapter, we will go over all these topics.
Most of the proof in this part are taken from [**42**] where the reader can find more information about analytic capacity.

1. Removable singularities

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We say that f has an isolated singularity at $z_0 \in \mathbb{C}$ if f is analytic in a deleted neighborhood Ω of z_0 but is not analytic at z_0. In this case, f has a Laurent expansion at z_0 given by

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \text{ for } z \in \Omega.$$

Recall that a deleted neighborhood Ω of z_0 (also called punctured disc in some books) is a set of the form $\Omega = \{z \in \mathbb{C}; 0 < |z - z_0| < \delta\}$.

We say that f has an isolated singularity at ∞ if f is analytic outside some bounded open set. It is clear that f has an isolated singularity at ∞ if and only if the function $z \to f\left(\dfrac{1}{z}\right)$ has an isolated singularity at 0.

Isolated singularities z_0 of f may be classified as follows (we will use the same notations as above):

The function f has a removable singularity at z_0 if there exists a functions g which is analytic at z_0 and such that $f(z) = g(z)$ for $z \in \Omega$. Obviously, this is equivalent to the fact that the Laurent expansion of f at z_0 is the power series

$$f(z) = \sum_{k=0}^{+\infty} a_k (z - z_0)^k \text{ for } z \in \Omega.$$

For instance, the function $z \to \dfrac{\sin z}{z}$ has a removable singularity at 0.

The point z_0 is a pole of order N ($N \in \mathbb{N}$) if f can be written as $f(z) = \dfrac{P(z)}{Q(z)}$ for $z \in \Omega$ where P and Q are analytic at z_0, $P(z_0) \neq 0$, and Q has a zero of order N at

H. Pajot: LNM 1799, pp. 67–79, 2002.

z_0. In this case, the Laurent expansion of f at z_0 is given by

$$f(z) = \sum_{k=-N}^{+\infty} a_k (z - z_0)^k \text{ for } z \in \Omega.$$

Finally, z_0 is an essential singularity if z_0 is neither a removable singularity nor a pole.

THEOREM 53 (Riemann's principle for removable singularities). *If f has an isolated singularity at z_0 and if f is bounded near z_0, then f has a removable singularity at z_0.*

See [**82**] for very interesting historical remarks about this result.

PROOF. Consider the Laurent expansion of f at z_0

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \text{ for } z \in \Omega.$$

By the Cauchy integral formula, the coefficients a_n are given (for r small enough) by

$$a_n = \frac{1}{2i\pi} \int_{\partial D(z_0,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Since f is bounded near z_0, there exist $r > 0$ and $M \geq 0$ such that $|f(z)| \leq M$ for $z \in \partial D(z_0, r)$. Thus, $|a_n| \leq \dfrac{1}{2\pi} \dfrac{M}{r^{n+1}} 2\pi r = \dfrac{M}{r^n}$. For $n < 0$, $\lim_{r \to 0} \dfrac{M}{r^n} = 0$, hence $a_n = 0$. □

This raises to the more general problem of determining (compact) sets of the complex plane which share with singletons the property of being removable singularities for bounded analytic functions.

2. The Painlevé Problem

Let $E \subset \mathbb{C}$ be a compact set. We say that E is removable for bounded analytic functions if, whenever U is an open set containing E, every bounded analytic function $f : U \setminus E \to \mathbb{C}$ has an analytic extension to U.
Example: It follows from Riemann's theorem for removable singularities that every singleton $\{z_0\}$ is removable.

The classical Painlevé problem consists in giving a geometric/metric characterization of such removable sets.

To study this problem, L. Ahlfors [**1**] defined the analytic capacity $\gamma(E)$ of E by

$$\gamma(E) = \sup\{|f'(\infty)|, f : \mathbb{C} \setminus E \to \mathbb{C} \text{ is analytic}, ||f||_\infty \leq 1\}$$

where $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$.
See [**16**] for an account of the work of L. Ahlfors related to the Painlevé problem.
Since f is analytic and bounded outside a compact set, by the discussion of the previous section, its Laurent series at ∞ can be written as

$$f(z) = a_0 + \frac{a_1}{z} + \dots\dots\dots + \frac{a_k}{z^k} + \dots.$$

and therefore, $a_1 = f'(\infty)$. This implies that $f'(\infty) = \dfrac{1}{2i\pi}\displaystyle\int_\Gamma f(z)dz$ where Γ is any curve in $\mathbb{C}$ separating E and ∞. Roughly speaking, $\gamma(E)$ measures the size of the unit ball of the space of bounded analytic functions outside E.
By considering $g(z) = \dfrac{f(z) - f(\infty)}{1 - \overline{f(\infty)}f(z)}$, it is not difficult to see that we can restrict in the definition of analytic capacity to functions vanishing at ∞ (for this, recall that, if $\alpha \in D(0,1)$, the Möbius transformation $M_\alpha(z) = \dfrac{z-\alpha}{1-\overline{\alpha}z}$ is an analytic automorphism of the unit disk).
The analytic capacity satisfies the following basic properties.
(i) If E, F are compact sets in $\mathbb{C}$ with $E \subset F$ then $\gamma(E) \leq \gamma(F)$.
(ii) Let a, $b \in \mathbb{C}$. If E is a compact in $\mathbb{C}$, then $\gamma(aE+b) = |a|\gamma(E)$.

There exists an extremal function for the analytic capacity, that is there exists a bounded analytic function $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $||f||_\infty \leq 1$ such that $f(\infty) = 0$ and $f'(\infty) = \gamma(E)$. This can be seen by using a normal family argument involving Montel's theorem (see for instance the discussion in [**40**] pages 306-309). It turns out that this extremal function is unique and is called the Ahlfors function of E. If E has n components, more can be said. For instance, the Ahlfors function f is analytic across $\partial\Omega$ (where $\Omega = \mathbb{C} \setminus E$) and is equal to 1 on $\partial\Omega$ (if $\partial\Omega$ is smooth). Moreover, f has n zeros on Ω (see [**1**], [**42**]).

If μ is a finite complex measure supported on E, then its Cauchy transform $\mathcal{C}_\mu$ defined by

$$\mathcal{C}_\mu(z) = \int \frac{d\mu(\xi)}{\xi - z} (\text{ for } z \notin E)$$

is analytic outside E. To see this, prove that $\overline{\partial} f = \pi\mu$ in the sense of distribution where $\overline{\partial} f = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right)$. Moreover, $\mathcal{C}_\mu(\infty) = 0$ and $\mathcal{C}'_\mu(\infty) = -\displaystyle\int d\mu$. This motivates the introduction of an other "analytic capacity":

$\gamma_+(E) = \sup\{||\mu||; \mu$ is a positive Radon measure supported on E such that $||\mathcal{C}_\mu||_\infty \leq 1\}$
(Here, $||\mu||$ denotes the total mass of μ. In other words, $||\mu|| = \mu(\mathbb{C})$).
By the previous remark, $\gamma_+(E) \leq \gamma(E)$ for every compact set $E \subset \mathbb{C}$. We will see in the last chapter that the converse holds, that is, there exists $C \geq 1$ such that $\gamma(E) \leq C\gamma_+(E)$ for every compact set $E \subset \mathbb{C}$. This solves a lot of problems concerning analytic capacity (see again the last chapter).

Remark. For general sets $E \subset \mathbb{C}$, it is not true that any bounded analytic function $f : \mathbb{C} \setminus E \to \mathbb{C}$ is the Cauchy transform of a complex measure μ supported on E (see the discussion in chapter III of [**42**]). However, we will see later that it is always true if the 1-dimensional Hausdorff measure of E is finite (see chapter 6).

PROPOSITION 54. *Let $E \subset \mathbb{C}$ be a compact set. The following assertions are equivalent.*
(i) $\gamma(E) = 0$;
(ii) Every bounded analytic function $f : \mathbb{C} \setminus E \to \mathbb{C}$ is constant;
(iii) E is removable for bounded analytic functions.

PROOF. It is clear that (ii) implies (i). Conversely, assume that (ii) fails. Then, there exists a bounded analytic function $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $f(\infty) = 0$ and $f(z_0) \neq 0$ for some $z_0 \in \mathbb{C} \setminus E$. Consider

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \text{ and } z \in \mathbb{C} \setminus E \\ f'(z_0) & \text{if } z = z_0. \end{cases} \tag{41}$$

It is not difficult to see that g is bounded and analytic in $\mathbb{C} \setminus E$ and satisfies $g'(\infty) = f(z_0) \neq 0$. Thus, $\gamma(E) > 0$.

The fact that (iii) implies (ii) follows from Liouville's theorem. On the other hand, assume that E satisfies (ii). We claim that E is totally disconnected. If not, the Riemann mapping theorem applied to $\mathbb{C} \setminus E_0$ where E_0 is one component of E produces a non constant bounded analytic function (see the next section for more details). Consider now an open set U containing E and a bounded analytic function f in $U \setminus E$. Fix z_0 in $U \setminus E$. Since E is totally disconnected, we can find two disjoint smooth Jordan curves Γ_1 and Γ_2 in U both surrounding E such that z_0 is inside Γ_1 and outside Γ_2. By the Cauchy integral formula,

$$f(z) = \frac{1}{2i\pi} \int_{\Gamma_1} \frac{f(z)}{z - z_0} dz - \frac{1}{2i\pi} \int_{\Gamma_2} \frac{f(z)}{z - z_0} dz.$$

We set, for $j = 1, 2$,

$$f_j(z) = \frac{1}{2i\pi} \int_{\Gamma_j} \frac{f(z)}{z - z_0} dz.$$

Note that these formulas can be used to define f_1 for $z \in U$ and f_2 for $z \in \mathbb{C} \setminus E$ since they do not depend on the choice of Γ_1 and Γ_2. It is clear that f_2 is bounded analytic in $\mathbb{C} \setminus E$. Thus, by hypothesis, f_2 is constant. But, $f_2(\infty) = 0$. Hence, f_2 is identically equal to 0 in $\mathbb{C} \setminus E$. Moreover, f_1 is analytic in U. Therefore, f_1 is an analytic extension of f to U (since $f(z) = f_1(z) + f_2(z) = f_1(z)$ for every $z \in U$). □

The proposition above does not provide a geometric characterization of removable sets. Moreover, the study of the analytic capacity is not so easy. For instance, it was a longstanding open problem to determine if γ is semi-additive, that is if there exists $C > 0$ such that $\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F))$ whenever E and F are disjoint compact sets in $\mathbb{C}$ (see the last chapter for a solution).

Remark. Analytic capacity plays an important role in the theory of rational approximation. For instance, let E be a compact set in $\mathbb{C}$. Let $R(E)$ be the algebra of functions f which can be uniformly approximated on E by rational functions with poles off E and let $C(E)$ be the space of continuous functions on E . Then, $C(E) = R(E)$ if and only if $\gamma(D \setminus E) = r$ for each disc D of radius r. For more details, see the nice survey [**104**] and the references therein.

3. Some examples

In this section, we will describe two useful methods to estimate the analytic capacity. The first one uses the theory of conformal mappings. The second one consists in exhibiting an appropriate measure μ supported on the set E such that the Cauchy transform $\frac{1}{z} * \mu$ is bounded outside E.

3.1. Continua in the complex plane. In this part, we will prove that the analytic capacity of a (non degenerate) continuum of the complex plane is not zero. More precisely, we have

THEOREM 55. *Let E be a non degenerate continuum of the complex plane $\mathbb{C}$. Then, $\gamma(E) \geq \frac{1}{4} diam E$.*

Recall that a non degenerate continuum of $\mathbb{C}$ is a compact connected set which is not a point. The proof is based on the Riemann mapping theorem and some distortion results for analytic functions (namely the Schwarz lemma and the one-quarter Koebe theorem). We first state these very important results.

Denote by $S_\infty = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. A region $\Omega \subset \mathbb{C}$ is simply connected if its complement in S_∞ is connected. We follow the terminology of [**2**].

THEOREM 56 (Riemann Mapping Theorem). *Let $\Omega \subset \mathbb{C}$ be a simply connected region which is not a point and the whole plane, and let z_0 be a point in Ω. Then, there exists a unique analytic function f in Ω normalized by the condition $f(z_0) = 0$ and $f'(z_0) > 0$ such that f defines a one-to-one mapping of Ω onto the unit disk $D(0,1) = \{w \in \mathbb{C}; |w| < 1\}$.*

Note that the Riemann mapping theorem can be extended to functions defined on the Riemann Sphere by the use of Möbius transformations (see [**40**] pages 294-297).

THEOREM 57 (Schwarz Lemma). *If f is analytic in the unit disk $D(0,1) = \{z \in \mathbb{C}; |z| < 1\}$ and satisfies the conditions*
(i) $|f(z)| \leq 1$ for every $z \in D(0,1)$;
(ii) $f(0) = 0$,
Then, $|f(z)| \leq |z|$ for every $z \in D(0,1)$, and $|f'(0)| \leq 1$. Moreover, if $|f(z)| = |z|$ for some $z \in D(0,1)$ or if $|f'(0)| = 1$, then there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$ for every $z \in D(0,1)$.

The Schwarz lemma follows from the maximum principle. For a proof, see for instance [**2**].

THEOREM 58 (Koebe's One-quarter Theorem). *If f is a one-to-one analytic function on the unit disk $D(0,1) = \{z \in \mathbb{C}; |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$, then $f(D(0,1)) \supset D(0, \frac{1}{4})$.*

By considering the function $f(z) = \dfrac{z}{(1-z)^2}$, you can easily see that the constant $\frac{1}{4}$ is sharp. See [**93**] for a proof and applications to potential theory.

We now prove that $\gamma(E) \geq \frac{1}{4} \mathrm{diam} E$ whenever E is a non degenerate continuum in $\mathbb{C}$.

PROOF. By the Riemann Mapping Theorem, there exists a unique conformal mapping g of $S_\infty \setminus E$ onto the unit disk such that $g(\infty) = 0$ and $g'(\infty) > 0$. We claim that $g'(\infty) = \gamma(E)$ (in other words, g is the Ahlfors function of E). To see this, first note that, by definition of the analytic capacity, $g'(\infty) \leq \gamma(E)$. On the other hand, consider an analytic function $f : \mathbb{C} \setminus E \to \mathbb{C}$ such that $||f||_\infty \leq 1$ and $f(\infty) = 0$.

Then, $F = f \circ g^{-1}$ satisfies the hypothesis of the Schwarz lemma and thus, $|F'(0)| \leq 1$. But, since $F'(0) = \dfrac{f'(\infty)}{g'(\infty)}$, we get $\gamma(E) \leq g'(\infty)$, thus $g'(\infty) = \gamma(E)$.

Let $z_0 \in E$ and consider $f(w) = \dfrac{\gamma(E)}{g^{-1}(w) - z_0}$ for $w \in D(0,1)$. Then, f is univalent and satisfies $f(0) = 0$, $f'(0) = 1$. Thus, by the Koebe's theorem, $f(D(0,1)) \supset D(0, \frac{1}{4})$. Hence, $(*) \dfrac{\gamma(E)}{|z - z_0|} \geq \dfrac{1}{4}$ whenever $z \in E$. Indeed, if $\dfrac{\gamma(E)}{|z - z_0|} < \dfrac{1}{4}$ for some $z \in E$, then $\dfrac{\gamma(E)}{z - z_0} \in f(D(0,1))$, that is there exists $w \in D(0,1)$ such that $\dfrac{\gamma(E)}{z - z_0} = \dfrac{\gamma(E)}{g^{-1}(w) - z_0}$. This implies that $z = g^{-1}(w)$ which is impossible since $z \in E$ and g is defined on $S_\infty \setminus E$. From $(*)$, we get $\gamma(E) \geq \frac{1}{4}\mathrm{diam}E$. □

We proved that, if E is connected, $\gamma(E) = g'(\infty)$ where g is the unique conformal mapping with the normalization $g(\infty) = 0$ and $g'(\infty) > 0$. We now give two applications of this result.

PROPOSITION 59. *$\gamma(D(z,r)) = r$ whenever $z \in \mathbb{C}$ and $r > 0$.*

This implies that, if $E \subset \mathbb{C}$ is a compact set, $\gamma(E) \leq \mathrm{diam}E$ since E is contained in a disk of radius $\mathrm{diam}E$. We left the proof of the proposition to the reader as an easy exercise.

PROPOSITION 60. *If $I \subset \mathbb{R}$ is an interval, then $\gamma(I) = \frac{1}{4}\mathcal{L}^1(I)$ (where $\mathcal{L}^1$ denotes the 1-dimensional Lebesgue measure).*

PROOF. We first consider the case of $I = [-2,2]$. For this, define $G(z) = z + \dfrac{1}{z}$. Then, $G : D(0,1) \to S_\infty \setminus E$ is analytic, one-to-one and satisfies $G(0) = \infty$. Consider now $g = G^{-1}$. Thus, $\gamma(E) = g'(\infty) = 1$.
If $I = [a,b]$ where $a, b \in \mathbb{R}$ with $a < b$, then

$$\gamma(I) = \gamma\left(\frac{b-a}{4}[-2,2] + \left(\frac{a+b}{2}\right)\right) = \frac{b-a}{4}\gamma([-2,2]) = \frac{b-a}{4} = \frac{1}{4}\mathcal{L}^1(I).$$

□

3.2. Analytic capacity of subsets of the real line. We have seen in the previous section that $\gamma(I) = \frac{1}{4}\mathcal{L}^1(I)$ if $I \subset \mathbb{R}$ is an interval. In this part, we will prove that this equality is true for every subset of the real line.

THEOREM 61. *Let $E \subset \mathbb{R}$. Then, $\gamma(E) = \frac{1}{4}\mathcal{L}^1(E)$.*

Remark. The previous theorem improves a result of L. Ahlfors and A. Beurling [**4**] who have observed that if $E \subset \mathbb{R}$,

$$\frac{\mathcal{L}^1(E)}{4} \leq \gamma(E) \leq \frac{\mathcal{L}^1(E)}{\pi}.$$

PROOF. We first give a theorem of Pommerenke which is a consequence of the fact that the Ahlfors function of a n-connected domain is a n-covering map.

THEOREM 62. *Let E_1, E_2,........, E_n be pairwise disjoint connected domains in* $\mathbb{C}$. *Then,* $\gamma\left(\cup_{i=1}^n E_i\right) \leq \gamma\left(E_1 + ... + E_n\right)$.

Assume first that $E \subset \mathbb{R}$ is a finite union of pairwise disjoint intervals E_1, ..., E_n. Then, $E_1 + ... + E_n$ is an interval of length $\mathcal{L}^1(E)$ and therefore, by Pommerenke's result, $\gamma(E) \leq \frac{1}{4}\mathcal{L}^1(E)$. The general case follows by approximating a compact set $E \subset \mathbb{R}$ by finite unions of disjoint intervals.

Conversely, let E be a compact set of the real line and consider $f(z) = \frac{1}{2}\int_E \frac{dt}{t-z}$ for $z \notin E$. Then, $f(\infty) = 0$ and $f'(\infty) = -\frac{\mathcal{L}^1(E)}{2}$.
Easy computations show that

$$|\mathrm{Im}\ f(z)| = \frac{|y|}{2}\int_E \frac{dt}{(t-x)^2+y^2} \leq \frac{1}{2}\int_{-\infty}^{+\infty}\frac{du}{1+u^2} = \frac{\pi}{2}.$$

Thus, Re $e^{f(z)} > 0$. Therefore, the function $F(z) = \frac{1-e^{f(z)}}{1+e^{f(z)}}$ satisfies $|F(z)| \leq 1$ and its Laurent expansion is of the form

$$F(z) = \frac{\mathcal{L}^1(E)}{4z} + \frac{a_2}{z^2} + ...$$

Hence, $\gamma(E) \geq \frac{\mathcal{L}^1(E)}{4}$. □

3.3. Analytic capacity and area. In this part, we will prove that a compact set $E \subset \mathbb{C}$ with non zero area is not removable for bounded analytic functions. More precisely, we have

THEOREM 63. *Let $E \subset \mathbb{C}$ with finite area. Then,* $\gamma(E) \geq \frac{1}{2}\sqrt{\frac{\textit{Area } E}{\pi}}$.

Remark. If $E \subset \mathbb{C}$ is analytic, then the previous estimate can be improved by $\gamma(E) \geq \sqrt{\frac{\text{Area } E}{\pi}}$ and it is sharp for a disc (see [**42**]).

PROOF. Consider $F(z) = \int\int_E \frac{dw_1 dw_2}{w-z}$ where $w = w_1 + iw_2$. Then, F is analytic outside E, $\lim_{z\to\infty} F(z) = 0$ and $\lim_{z\to\infty} zF(z) = -\text{Area } E$.
Write now the Laurent expansion of F at ∞:

$$F(z) = \frac{-\text{Area } E}{z} + \frac{a_2}{z^2} + ...$$

Consider the disc $\Delta = D(z, R)$ where R is given by the relation $\pi R^2 =$ Area E.
Then,

$$|F(z)| \leq \int\int_E \frac{dw_1 dw_2}{|w-z|} \leq \int\int_{E\cap\Delta}\frac{dw_1 dw_2}{|w-z|} + \int\int_{E\setminus\Delta}\frac{dw_1 dw_2}{|w-z|}.$$

It is not difficult to see that

$$\int\int_{E\setminus\Delta}\frac{dw_1 dw_2}{|w-z|} \leq \int\int_{\Delta\setminus E}\frac{dw_1 dw_2}{|w-z|}.$$

Thus, $F(z) \leq \int\int_{\Delta} \frac{dw_1 dw_2}{|w-z|} = 2\pi R$. Consider $g(z) = \frac{f(z)}{2\pi R}$ for $z \notin E$. By the previous computations, $||g||_\infty \leq 1$, $g(\infty) = 0$ and $|g'(\infty)| = \frac{\text{Area } E}{2\pi R} = \frac{R}{2} = \frac{1}{2}\sqrt{\frac{\text{Area} E}{\pi}}$. Hence, $\gamma(E) \geq \frac{1}{2}\sqrt{\frac{\text{Area } E}{\pi}}$.

□

4. Analytic capacity and metric size of sets

We now come back to the Painlevé problem. The examples we have seen in the previous section indicate that the metric size of the set should play a role. It turns out that this intuition can be expressed in terms of Hausdorff measure and Hausdorff dimension.

THEOREM 64. *Let E be a compact of the complex plane $\mathbb{C}$.*

(A) If $H^1(E) = 0$, then $\gamma(E) = 0$.

(B) If $dim_H(E) > 1$, then $\gamma(E) > 0$.

Here, H^1 denotes the 1-dimensional Hausdorff measure, and $\dim_H E$ is the Hausdorff dimension of E (see the first chapter).

PROOF. (A) Let $\varepsilon > 0$. Since $H^1(E) = 0$, the set E can be covered by balls such that the sum of diameters of these balls is less than ε. Hence, we can surround E by a finite collection of C^1 curves Γ_i such that $\sum_i l(\Gamma_j) \leq 2\pi\varepsilon$. Let $f : \mathbb{C} \setminus E \to \mathbb{C}$ be a bounded analytic function. Assume that $f(\infty) = 0$. Therefore, for any z outside the domain bounded by the curves Γ_j, the Cauchy integral formula yields

$$f(z) = \frac{1}{2\pi} \sum_j \int_{\Gamma_j} \frac{f(\xi)}{\xi - z} d\xi.$$

Thus,

$$|f'(\infty)| = \frac{1}{2\pi} \left| \sum_j \int_{\Gamma_j} f(\xi) d\xi \right| \leq \frac{1}{2\pi} ||f||_\infty \sum_j l(\Gamma_j) \leq \varepsilon ||f||_\infty.$$

By taking $\varepsilon \to 0$, we get $\gamma(E) = 0$.

Remark. Let E be a compact set in $\mathbb{C}$. We define its Painlevé length by

$$PL(E) = \inf l(\partial V)$$

where the infimum is taken over open sets $V \supset E$ such that ∂V consists of finitely many analytic Jordan curves surrounding E (here $l(\partial V)$ is the usual length). Then, a slight modification of our argument above shows that $\gamma(E) \leq \frac{PL(E)}{2\pi}$ (see [**42**]).

(B) Fix d so that $1 < d < \dim_H E$. Then, $H^d(E) = +\infty$.
By Frostman Lemma, there exists a finite positive Radon measure μ supported on E

such that $\mu(B(z,r)) \le r^d$ whenever $z \in \mathbb{C}$ and $r > 0$. Consider $f = \frac{1}{z} * \mu$. We have seen before that f is analytic outside E. Moreover,

$$\begin{aligned} |f(z)| &\le \frac{1}{2\pi}\int_{|\xi - z|\ge 1} 1 d\mu(\xi) + \sum_j \frac{1}{2\pi}\int_{2^{-j-1}<|\xi-z|\le 2^{-j}} |z-\xi|^{-1} d\mu(\xi) \\ &\le \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{j=0}^{\infty} 2^{j+1}\mu(B(z,2^{-j})) \\ &\le C\left(1 + \sum_{j=0}^{+\infty} 2^j 2^{-jd}\right) < \infty. \end{aligned}$$

Therefore, f is bounded outside E and $f'(\infty) = -\mu(\mathbb{C}) \neq 0$. Thus, $\gamma(E) > 0$.

□

We will see in the next section that there exist compact sets E in $\mathbb{C}$ such that $H^1(E) > 0$ but $\gamma(E) = 0$.

Remark. In fact, a purely metric characterization of removable sets for bounded analytic functions (that is only in terms of Hausdorff measures or Hausdorff dimension) is impossible. See the discussion in [**15**].

5. Garnett-Ivanov's counterexample

Let $E \subset \mathbb{C}$ be the linear four corner Cantor set as described before proposition 7. Recall that $E = \cap_{j=0}^{\infty} E_n$ where E_n is the union of 4^n squares denoted by Q_n^j, $j = 1, 2, ..., n$ whose side length is 4^{-n}. We have seen that $H^1(E) > 0$.

Theorem 65. *Let E be the linear four corners Cantor set. Then, $\gamma(E) > 0$*

The original proof is due independently to J. Garnett [**43**] and L. Ivanov. Here, we present the proof of P. Jones [**49**] (as described in [**17**]). See also [**76**] for an other proof.

The first example of removable compact set (in the plane) but with non zero length is due to A. G. Vitushkin (see for instance [**42**] for a description of this set).

Proof. P. Jones'argument is based on some estimates of the harmonic measure which we first recall.

Fix $z \in \mathbb{C} \setminus E$ such that $d(z, E) \le 1$. Let $n \in \mathbb{N}$ such that $C^{-1}4^{-n} \le d(z, E) \le C4^{-n}$ and let $Q(z) \in E_n$ such that $C^{-1}4^{-n} \le d(z, Q(z)) \le C4^{-n}$.

We denote by ω_z the harmonic measure relative to z and $G(z, .)$ is the Green's function for $\mathbb{C}$ with pole at z (see [**93**] for definitions)).

Lemma 66. *There exists $C > 0$ such that*

(i) $C^{-1} \le \omega_z(Q(z)) \le C$;

(ii) If $d(z, E) \le \frac{1}{2} d(\xi, E) \le 1$, then $C^{-1}G(z, \xi) \le \omega_\xi(Q(z)) \le CG(z, \xi)$.

See [**17**] for a proof.

Consider now $f : \overline{\mathbb{C}} \setminus E \to \mathbb{C}$ a bounded analytic function and $\xi \in \mathbb{C} \setminus E$. The Green's formula gives (see again [**17**] for details)

$$\frac{1}{2}\int_E |f(z) - f(\xi)|^2 d\omega_\xi(z) = \int_{\mathbb{C}\setminus E} |f'(z)|^2 G(z,\xi) d\mathcal{L}^2(z). \tag{42}$$

Thus, since f is bounded and ω_z is a probability measure, this yields

$$(*)\quad \sup_{\xi\in\mathbb{C}\setminus E}\int_{\mathbb{C}\setminus E}|f'(z)|^2G(z,\xi)d\mathcal{L}^2(z)<\infty.$$

Remark. The estimate (42) is classical in the theory of BMO functions on the unit disc (see [**44**]). By analogy of this case, we will say that an analytic function $f:\mathbb{C}\setminus E\to\mathbb{C}$ belongs to $BMO(\mathbb{C}\setminus E)$ if (*) holds for f (here, E is any compact set in the complex plane). L. Carleson asked if $BMO(\mathbb{C}\setminus E)$ is trivial if and only if E is removable for bounded analytic functions. P. Jones' proof of theorem 65 gives some credence to this conjecture.

We now go through the proof of the theorem. First, construct circles C_n^j, $j=1,...,4^n$, such that

(i) C_n^j and Q_n^j have the same center;
(ii) $C4^{-n}\le d(Q_n^j,C_n^j)\le C4^n$;
(iii) If $m\ge n$ and $Q_m^i\subset Q_n^j$, then C_n^j surrounds C_m^i;
(iv) $d(C_m^i,C_n^j)\ge C^{-1}4^{-n}$ if $n\le m$.

Set $A_n^j=\{z\in\mathbb{C};d(z,C_n^j)\le\varepsilon\}$ where $\varepsilon>0$ is small enough such that $d(A_n^i,A_m^j)\ge C^{-1}4^{-n}$ if $n\le m$ and $n\ne m$.

Let $f:\mathbb{C}\setminus E\to\mathbb{C}$ be a bounded analytic function with $f(\infty)=0$. Our goal is to show that $f(z)=0$ if $z\in\mathbb{C}\setminus E$.
For this, set $D_n^j=\sup_{z\in C_n^j}4^{-n}|f'(z)|$. Note that, by the mean value theorem,

$$(D_n^j)^2\le C\int_{A_n^j}|f'(z)|^2d\mathcal{L}^2(z).$$

Fix now A_n^j and choose $\xi\notin E$ such that $C^{-1}4^{-n}\le d(\xi,Q_n^j)\le C4^{-n}$ and $d(\xi,Q)\ge 2d(z,Q_n^j)$ for any $z\in A_n^j$. Therefore, by lemma 66, we get for any $Q_m^i\subset Q_n^j$,

$$\begin{aligned}(D_m^i)^2\omega_z(Q_m^i) &\le C\int_{A_m^i}|f'(z)|^2\omega_z(Q_m^i)d\mathcal{L}^2(z)\\ &\le C\int_{A_m^i}|f'(z)|^2G(z,\xi)d\mathcal{L}^2(z).\end{aligned}$$

Thus, since the annuli A_m^i are disjoint,

$$\begin{aligned}(**)\sum_{Q_m^i\subset Q_n^j}(D_m^i)^2\omega_\xi(Q_m^i) &\le C\int_{\mathbb{C}\setminus E}|f'(z)|^2G(z,\xi)d\mathcal{L}^2(z)\\ &\le K_0\end{aligned}$$

where $K_0>0$ does not depend on Q_n^j ! This follows from (*).

LEMMA 67. *For every $\delta>0$, there exists $M=M(\delta)\in\mathbb{N}$ such that for every $n\in\mathbb{N}$, every $j=1,....,4^n$, there exists $Q_m^i\subset Q_n^j$ with $m\le n+M$ and $D_m^i\le\delta$.*

PROOF. Since $d(\xi,Q_n^j)$ is comparable to 4^{-n}, by lemma 66,

$$\sum_{Q_m^i\subset Q_n^j}\omega_\xi(Q_m^i)=\omega_\xi(Q_n^j)\le C.$$

Assume now that for every $m \leq n+M$, every $i = 1, ..., 4^m$, $D^i_m \geq \delta$. Then,

$$\sum_{Q^i_m \subset Q^j_n} (D^i_m)^2 \omega_\xi(Q^i_m) \geq C\delta^2 M >> K_0$$

if M is big enough. This contradicts (**) and the lemma is proved. □

Using this lemma, we can easily construct a finite collection of pairwise disjoint squares $(Q^{j_\alpha}_{n_\alpha})$ such that
(i) $D^{j_\alpha}_{n_\alpha} \leq \delta$ for any α;
(ii) $H^1(E \setminus \cup_\alpha Q^{j_\alpha}_{n_\alpha}) \leq \delta$.

We choose now a finite collection of pairwise disjoint squares $(Q^{i_\beta}_{m_\beta})$ which are also disjoint from all the $(Q^{j_\alpha}_{n_\alpha})$ such that
(i) $E \subset (\bigcup_\alpha Q^{j_\alpha}_{n_\alpha}) \cup (\bigcup_\beta Q^{j_\beta}_{m_\beta})$;
(ii) $\sum_\beta 4^{-m_\beta} \leq C\delta$.
This is possible to find such family since $H^1(E \setminus \cup_\alpha Q^{j_\alpha}_{n_\alpha}) \leq \delta$.
Fix $z \in \mathbb{C} \setminus E$. By the Cauchy formula, we get

$$f(z) = \frac{1}{2\pi} \sum_\alpha I_\alpha + \frac{1}{2\pi} \sum_\beta I_\beta$$

where $I_\alpha = \displaystyle\int_{C^{j_\alpha}_{n_\alpha}} \frac{f(\xi)}{\xi - z} d\xi$ and $I_\beta = \displaystyle\int_{C^{i_\beta}_{m_\beta}} \frac{f(\xi)}{\xi - z} d\xi$. Choose $z_\alpha \in C^{j_\alpha}_{n_\alpha}$ and write

$$\begin{aligned}
|I_\alpha| &= \left| \int_{C^{j_\alpha}_{n_\alpha}} \frac{f(z) - f(z_\alpha)}{\xi - z} d\xi \right| \\
&= \left| \int_{C^{j_\alpha}_{n_\alpha}} \left(\frac{f(z) - f(z_\alpha)}{z - z_\alpha} \right) \left(\frac{z - z_\alpha}{\xi - z} \right) d\xi \right| \\
&\leq C 4^{-2n_\alpha} \sup_{z \in C^{j_\alpha}_{n_\alpha}} |f'(z)|^2 \\
&\leq C 4^{-n_\alpha} D^{j_\alpha}_{n_\alpha} \\
&\leq C\delta 4^{-n_\alpha}.
\end{aligned}$$

Therefore, $\sum_\alpha |I_\alpha| \leq C\delta \sum_\alpha 4^{-n_\alpha} \leq C\delta H^1(E)$.
Moreover, $|I_\beta| \leq C(z)||f||_\infty 4^{-m_\beta}$ where $C(z) = d(z, \partial E)$. Therefore,

$$\begin{aligned}
\sum_\beta |I_\beta| &\leq C(z)||f||_\infty \sum_\beta 4^{-m_\beta} \\
&\leq C(z)||f||_\infty \, \delta.
\end{aligned}$$

Hence, there exists $C > 0$ depending only on E and on z such that $|f(z)| \leq C\delta$ for any $\delta > 0$. Thus, $f(z) = 0$.

□

Remark. J. Garnett and S. Yoshinobu [**46**] have observed that the same proof can be used to show that, if E is a generalized four corners Cantor set associated to a sequence (λ_n) with $\lambda_n = 4^{-n}(\log(n+1))^{\frac{1}{p}}$ for some $p > 2$, then $\gamma(E) = 0$. We will improve this result in the last chapter.

6. Who was Painlevé ?

Paul Painlevé was a famous French mathematician, but also a statesman.
He is born in Paris in 1863. Very gifted in literature and in sciences, he was admitted at the *Ecole Normale Supérieure* in 1883. Then, he received his *agrégation* in mathematics in 1886 and defended his thesis [**83**] whose subject was the study of singularities of analytic functions with applications to differential equations in 1887. The referee of his thesis was E. Picard (the report can be found in [**47**] page 342) and the president of the committee was C. Hermitte. P. Painlevé became Associate professor at the University of Lille in 1887, and moved in 1892 to Paris where he taught at the *Faculté des Sciences (La Sorbonne)*, the *Ecole Polytechnique*, the *Collège de France* (1896) and the *Ecole Normale Supérieure*. He won the *Grand Prix des Sciences Mathématiques* in 1890 and was elected a member of the geometry section of the *Académie des Sciences* in 1900.

P. Painlevé was very interested in the very recent field of aviation: He was W. Wright's first French passenger in 1908, was professor at the *Ecole Supérieure d'Aeronautique* (1909), wrote with Emile Borel a book entitled "*L'Aviation*" (Edition Arcan, 1910) and headed several commissions on aerial navigation.

P. Painlevé entered in politics in 1906 when he was elected to the Chamber of Deputy from Paris (*Quartier Latin*). During the first world war, he was involved in the French political life. In 1914, he created the *Service des Inventions pour les Besoins de la Défense Nationale*. Brought into A. Briand's cabinet in October 1915, he was given the Education Ministry. In 1917, he was appointed War Minister in A. Ribot's new ministry and was *Président du Conseil* (Prime Minister) from September 1917 to November 1917. He supported the efforts of the army of the Near East in the hope of detaching Austria-Hungary from the German Alliance (He was a fervent "Easterner"). He conducted the negotiations with W. Wilson over the sending of American troops to France. He made the decision to replace General Nivelle with P. Petain as commander of the Verdun salient and appointed *Maréchal* Foch as head of the Allied chiefs of staff before resigning in favor of G. Clemenceau.

In 1920, P. Painlevé was commissioned by the Chinese government to reorganize the country's railroads. He led the *Ligue de la République* from 1921 to 1922 before joining the *Cartel des Gauches* (the Left).
He served again as *Président du Conseil* from April-November 1925 but, like his predecessor E. Herriot, was unable to secure passage of the *Cartel*'s economic program. As War Minister (1925-1929), he sponsored the construction of the *Ligne Maginot* which was supposed to provide a defense against Germany in the next war (in fact, this *Ligne* was never used). He inaugurated the Air Ministry in 1930 serving as its chief twice until his retirement in 1932.

He died in Paris on 29 October 1933 at the age of 69.

P. Painlevé was interested by problems in pure and applied mathematics: study of singularities of analytic functions and of singular points of differential equations, rational transformations of algebraic curves, n-body problem, problems of friction,.... See [**84**] for an account of his work before 1900.

One motivation of his thesis [**83**] was the study of the nature of singularities of single valued (but also multi valued) analytic functions by using the recent set theory developed by G. Cantor. For this, he classified disconnected perfect sets E of the complex plane as follows (compare with the notion of 1-dimensional measure and of Painlevé length we have seen previously).
Let $\varepsilon > 0$. First, we surround the points of E by finitely many curves α_1,..., α_p such that their interiors are disjoint and any curve α_i is contained in a disk with radius ε. Denote by L_ε the sum of the perimeters of these curves. The set E is said to be
(i) "*ponctuel*" if L_ε tends to 0 when $\varepsilon \to 0$ (for a suitable choice of the α_i's);
(ii) "*Semi-linéaire*" if (i) does not hold but L_ε is bounded when $\varepsilon \to 0$ (for a suitable choice of the α_i's);
(iii) "*Semi-superficiel*" if L_ε tends to ∞ with $\dfrac{1}{\varepsilon}$ for any choice of the α_i's.
We follow here the terminology and the presentation given in [**84**]. Using this notion of "*ponctuel*" set, Painlevé stated (without a complete proof) an analog of theorem 64 part (A) (see [**83**] page 24). This explains partially where the name "Painlevé Problem" comes from.
It should be noted that a proof of theorem 64 part (A) can be found in [**6**] without any mention to Painlevé's work.

CHAPTER 6

The Denjoy and Vitushkin conjectures

In this chapter, we will give a partial solution to the Painlevé problem. Indeed, we will prove (at least in the Ahlfors regular case) that, if $E \subset \mathbb{C}$ is a compact set with $H^1(E) < +\infty$, then $\gamma(E) = 0$ if and only if E is purely unrectifiable. We will also construct a set $E \subset \mathbb{C}$ with $H^1(E) = +\infty$ such that $\gamma(E) > 0$ but its projections in almost all directions have zero length.

1. The statements

The main result of this chapter is

THEOREM 68. *Let $E \subset \mathbb{C}$ be a compact set with $H^1(E) < +\infty$. Then E is removable for bounded analytic functions if and only if E is purely 1-unrectifiable.*

In the sequel, we will say that E has finite length if $H^1(E) < +\infty$. Recall that E is removable for bounded analytic functions if and only if $\gamma(E) = 0$ (where $\gamma(E)$ denotes the analytic capacity of E) and that E is purely 1-unrectifiable if $H^1(E \cap \Gamma) = 0$ for every rectifiable curve $\Gamma \subset \mathbb{C}$.

For some historical reasons, we prefer to split the previous theorem into two parts.

THEOREM 69. *Let $E \subset \mathbb{C}$ be a subset of a rectifiable curve Γ. Then, $\gamma(E) = 0$ if and only if $H^1(E) = 0$.*

The direction $\Leftarrow$ has already been proved (and is still true, even if the set is not a subset of a rectifiable curve). The other direction was proved by A. Denjoy. Unfortunately, there was a gap in A. Denjoy's argument, it is why this result was called the Denjoy conjecture. The proof was completed only after the proof by A. P. Calderon of the L^2 boundedness of the Cauchy singular operator on Lipschitz graphs with small constant [**11**]. In the second section, we will present the proof of the Denjoy conjecture given in [**17**] and sketch an alternative proof using the Garabedian formula. The first proof is based on a duality argument that we will explain in the first section. Note that Garabedian formula follows also from a (different) duality argument. Theorem 69 implies the direct sense of theorem 68. Indeed, if E is not purely unrectifiable, then there exists a rectifiable curve Γ such that $H^1(\Gamma \cap E) \neq 0$. Hence, by theorem 69, $\gamma(E \cap \Gamma) > 0$, and thus $\gamma(E) > 0$.

THEOREM 70. *Let $E \subset \mathbb{C}$ be a compact set with finite length. If $\gamma(E) > 0$, then there exists a rectifiable curve Γ in $\mathbb{C}$ such that $H^1(E \cap \Gamma) \neq 0$.*

Obviously, theorems 69 and 70 give theorem 68.

Let $E \subset \mathbb{C}$. Its Favard length (denoted by $\mathrm{Fav}(E)$) is defined by

$$\mathrm{Fav}(E) = \frac{1}{\pi} \int_0^\pi H^1(P_\theta(E)) d\theta$$

H. Pajot: LNM 1799, pp. 81–103, 2002.

where P_θ is the projection on the line passing through the origin of $\mathbb{C}$ and making an angle θ with the real axis. We have seen in the first chapter (see theorem 17) that, if E has finite length, E is purely unrectifiable if and only if $\text{Fav}(E) = 0$. Therefore, theorem 68 can be restated as follows.

Theorem 68 (revisited). *Let $E \subset \mathbb{C}$ such that $H^1(E) < +\infty$. Then, $\gamma(E) = 0$ if and only if $Fav(E) = 0$.*
This result was initially conjectured by A. Vituskhin without assuming that E has finite length. We will see in the last section of this chapter that there exist compact sets $E \subset \mathbb{C}$ with $\text{Fav}(E) = 0$, but $\gamma(E) > 0$ (Of course, these sets have infinite length !). Nevertheless, before the proof of theorem 70 by G. David [**26**], the result of theorem 68 was usually called the Vitushkin conjecture.

Remarks. 1) In 2000, after a talk I gave at the University of Washington, D. Marshall told me that W. Hayman was the first to guess that Vitushkin conjecture should be true for sets with finite length. But, I prefer here to follow the standard terminology.
2) We should emphasize again on that fact that rectifiability properties of sets with infinite length are not well understood. However, it should be noticed that a version of theorem 17 in this general setting is given in [**81**].

Theorem 70 was first proved for Ahlfors regular sets by P. Mattila, M. Melnikov and J. Verdera [**69**]. We will present their proof in the third section and explain how to modify it in the general case. For this, we will use a lot of notions and results we have seen in the previous chapters.

2. The standard duality argument

Let X be a locally compact Hausdorff space. Denote by $C_0(X)$ the space of continuous functions on X vanishing at infinity and by $\mathcal{M}(X)$ the space of complex finite Radon measures on X. For any bounded linear operator $T : \mathcal{M}(X) \to C_0(X)$, denote by T^* its transpose. Recall that $T^* : \mathcal{M}(X) \to C_0(X)$ is such that for ν, $\lambda \in \mathcal{M}(X)$,

$$\int (T\nu)(x) d\lambda(x) = \int (T^*\lambda)(y) d\nu(y).$$

Theorem 71. *Let X be a locally compact Hausdorff space, μ a positive Radon measure and $T : \mathcal{M}(X) \to C_0(X)$ a bounded linear operator. Assume that T^* is bounded. The following statements are equivalent.*

(i) *The operator T^* is of weak type $(1,1)$, that is there exists $C_1 > 0$ such that*

$$\mu(\{x \in X; |T^*\nu(x)| > \alpha\}) \le \frac{C_1}{\alpha} |\nu(X)|$$

whenever $\alpha > 0$ and $\nu \in \mathcal{M}(X)$.

(ii) *There exists $C_2 > 0$ such that for any Borel set $E \subset X$ with $0 < \mu(E) < +\infty$, $\mu(E) \le 2 \sup \int h d\mu$, where the supremum is taken over all μ-measurable functions $h : X \to [0,1]$ supported on E such that $|T(hd\mu)|(x) \le C_2$ whenever $x \in X$.*

(iii) *There exists* $C_3 > 0$ *such that for any compact set* $K \subset X$, $\mu(K) \le C_3 \sup \left| \int d\lambda \right|$ *where the supremum is taken over all measures* λ *supported on* K *such that* $|\lambda(X)| \le \mu(K)$ *and* $|T\lambda(x)| \le 1$ *for all* $x \in X$.

PROOF. We first prove the implication $(i) \Rightarrow (ii)$ following [**17**]. For this, assume that the conclusions of (ii) are false for some $E \subset X$ and consider

$$\begin{aligned}
\mathcal{B}_0 &= \left\{ f : X \to [0,1]; f = 0 \text{ almost everywhere on } X \setminus E, \int_E f d\mu \ge \frac{\mu(E)}{2} \right\}, \\
\mathcal{B}_1 &= \{T(fd\mu); f \in \mathcal{B}_0\}, \\
\mathcal{B}_2 &= \{g \in C_0(X); ||g||_\infty < C_2\}.
\end{aligned}$$

Then, $\mathcal{B}_1$ and $\mathcal{B}_2$ are two convex disjoint subsets of $C_0(X)$ (since (ii) is supposed to be false). By the Hahn-Banach theorem, there exists $l \in (C_0(X))^*$ (where $(C_0(X))^*$ is the dual space of $C_0(X)$) so that $\mathrm{Re}\, l(g) < \mathrm{Re}\, l(h)$ for any $g \in \mathcal{B}_2$, any $h \in \mathcal{B}_1$.
By the Riesz representation theorem, there exists a finite Radon measure λ such that

$$\mathrm{Re} \left(\int T(fd\mu) d\lambda \right) > \mathrm{Re} \int g d\lambda$$

for any $f \in \mathcal{B}_0$, any $g \in \mathcal{B}_2$.
By maximizing the right hand side, we get, for any $f \in \mathcal{B}_0$,

$$\mathrm{Re} \left(\int f(T^*\lambda) d\mu \right) > C_2 \lambda(X). \tag{43}$$

Let $\alpha = \dfrac{3C_1\lambda(X)}{\mu(E)}$. Since T^* is of weak type $(1,1)$ and by choice of α,

$$\mu\left(\{x \in X; |T^*\lambda(x)| > \alpha\}\right) \le \frac{C_1}{\alpha}\lambda(X) = \frac{\mu(E)}{3}.$$

Therefore,

$$\mu(\{x \in E; |T^*\lambda(x)| \le \alpha\}) \ge \frac{2}{3}\mu(E).$$

Hence, we can find a closed subset F of E such that $\mu(F) > \dfrac{1}{2}\mu(E)$ and $|T^*\lambda(x)| \le \alpha$ for any $x \in F$.
Consider $f = \chi_F$. Then, $f \in \mathcal{B}_0$ and

$$\left| \int f(x)(T^*\lambda(x)) d\mu(x) \right| \le \alpha\mu(F) \le 3C_1\lambda(X).$$

This contradicts (43) if $C_2 = 4C_1$ and the proof is complete.
The fact that $(ii) \Rightarrow (iii)$ is obvious. We now prove the implication $(iii) \Rightarrow (i)$. For this, set for $\beta > 0$ and $\theta \in [0, 2\pi[$, $E_{\beta,\theta} = \{x \in X, T^*\nu(x) \in D(\beta e^{i\theta}, C_3^{-1}\frac{\beta}{4})\}$. By using a covering of $\mathbb{C}$ by discs of the form $D(\beta e^{i\theta}, C_3^{-1}\dfrac{\beta}{4})$, we claim that it is enough to prove $\mu(E_{\beta,\theta}) \le C\dfrac{|\nu(X)|}{\beta}$. To prove the last estimate, consider a compact set $F_{\beta,\theta} \subset E_{\beta,\theta}$ such that $2\mu(F_{\beta,\theta}) \ge \mu(E_{\beta,\theta})$. By (iii), there exists $\lambda \in \mathcal{M}(X)$ supported on $F_{\beta,\theta}$ such

that $|\lambda(X)| \leq \mu(F_{\beta,\theta})$, $|T\lambda(x)| \leq 1$ for all $x \in X$ and $\mu(F_{\beta,\theta}) \leq 2C_3|\int d\lambda|$. Thus,

$$\begin{aligned}
\mu(F_{\beta,\theta}) &\leq 2C_3\beta^{-1}|\int_{F_{\beta,\theta}} \beta e^{i\theta} d\lambda| \\
&\leq 2C_3\beta^{-1}|\int_{F_{\beta,\theta}} T^*(\nu) d\lambda| + 2C_3\beta^{-1}|\int_{F_{\beta,\theta}} (\beta e^{i\theta} - T^*(\nu)) d\lambda| \\
&\leq 2C_3\beta^{-1}|\int T(\lambda) d\nu| + \frac{1}{2}|\lambda(X)| \\
&\leq 2C_3\beta^{-1}|\nu(X)| + \frac{1}{2}\mu(F_{\beta,\theta}).
\end{aligned}$$

Therefore, $\mu(F_{\beta,\theta}) \leq 4C_3\beta^{-1}|\nu(X)|$ and $\mu(E_{\beta,\theta}) \leq 2\mu(F_{\beta,\theta}) \leq 8C_3\beta^{-1}\nu(X)$.
Fix now $\alpha > 0$. To conclude, we consider a covering of $\{z \in \mathbb{C}; |z| > \alpha\}$ by some sets $E_{\beta,\theta}$. More precisely,

$$\{x \in X; |T^*\nu(x)| > \alpha\} \subset \bigcup_{n=0}^{+\infty} \bigcup_{k=1}^{q} E_{(1+\frac{1}{4C_3})^n\alpha, 2\pi\frac{k}{q}}$$

where q is the integral part of $10^3\ C_3$. Thus,

$$\begin{aligned}
\mu(\{x \in X; |T^*\nu(x)| > \alpha\}) &\leq 8C_3|\nu(X)| \sum_{n=0}^{+\infty} \sum_{k=1}^{q} \frac{1}{(1+\frac{1}{4C_3})^n\alpha} \\
&\leq \frac{C}{\alpha} C_3^3 |\nu(X)|.
\end{aligned}$$

Note that the constants C_1, C_2 and C_3 satisfy $A^{-1}C_1 \leq C_2 \leq AC_1$ and $A^{-1}C_1 \leq (C_3)^3 \leq AC_1$ for some constant $A > 0$. □

As far as I know, this type of duality argument was used for the first time in [**103**] (see also [**30**]).

We now explain briefly why theorem 71 is a useful tool to study analytic capacity. For this, assume that X is a subset of $\mathbb{C}$ and that there exists a positive Radon measure μ supported on X such that

(i) μ is doubling, that is $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ whenever $x \in \text{Supp}\mu$, $r \in (0, \text{diam}(\text{ Supp}\mu))$;
(ii) The Cauchy operator is bounded on $L^2(\mu)$.

Then, by theorem 48, the Cauchy operator is of weak type $(1, 1)$. Let now E be a Borel subset of X such that $0 < \mu(E) < +\infty$. By theorem 71 applied to the Cauchy operator on (X, μ), up to some technical problems, there exists a function $h : X \to [0, 1]$ supported on E such that $2\int h d\mu \geq \mu(E)$ and the Cauchy transform of $hd\mu$ (see chapter 5) is bounded. Thus, $\gamma^+(E) \geq C^{-1}\mu(E)$ for some absolute constant $C > 0$ and therefore $\gamma(E) \geq C^{-1}\mu(E)$.

3. Proof of the Denjoy conjecture

In this section, we will prove that, if E is a subset of a rectifiable curve with $H^1(E) > 0$, then $\gamma(E) > 0$ (theorem 69). For this, we first show that it is enough to prove theorem 69 in the special case where Γ is a Lipschitz graph. Then, we will use the strategy described at the end of the previous section to a Lipschitz graph endowed with its arc length measure.

THEOREM 72. *Let E be a subset of a rectifiable curve Γ of $\mathbb{C}$ such that $H^1(E) > 0$. Then, there exists a Lipschitz graph G in $\mathbb{C}$ such that $H^1(E \cap G) > 0$.*

PROOF. Without loss of generality, we may assume that Γ is (up to some rotation) the graph of a continuous function $g : \mathbb{R} \to \mathbb{R}$. To see this, recall that a rectifiable curve is rectifiable in the sense of geometric measure theory (see chapter 1). Thus, there exists $M_0 \in E \cap \Gamma$ such that Γ has an approximate tangent line (denoted by D_0) at M_0 and $\theta^1(M_0, \Gamma) = \theta^1(M_0, E) = 1$ (the last point follows from proposition 9). Let $\varepsilon > 0$ (to be chosen later) and let $R = R(\varepsilon) > 0$ such that

$$\begin{aligned} 2(1-\varepsilon)R &\leq H^1(B(M_0,R)\cap E) \leq 2(1+\varepsilon)R \\ 2(1-\varepsilon)R &\leq H^1(B(M_0,R)\cap \Gamma) \leq 2(1+\varepsilon)R \\ \varepsilon^{10}R &\leq H^1(\Gamma \setminus C(M_0,R,\varepsilon,D_0)). \end{aligned}$$

The cone $C(M_0, R, \varepsilon, D_0)$ has been defined in the first chapter (section 5). Up to some rotations and translations, we can assume that D_0 is the real line and that M_0 is the origin of the complex plane. For any $t \in (-R, +R)$, set $L_t = \{t + iy; y \in \mathbb{R}\}$ and let $p(t)$ be the point of $\Gamma \cap L_t$ which is the closest to t. If two such points exist, take $p(t)$ as the point located in the upper half plane. By smoothing the set $\{t + ip(t); t \in (-R, +R)\}$, we easily get the graph of a continuous function $g : (-R, +R) \to \mathbb{R}$ which contains a "big piece" of E (if ε is small enough). Thus, up to some rotation, we can assume that in the neighborhood of M_0, Γ is of the form $\Gamma_0 = \{x + if(x), x \in I\}$ where f is continuous and of bounded variation on $I = [a, b]$ (since Γ is rectifiable). Recall that this means that

$$F = \sup \left(\sum_{j=1}^{N} |f(t_j) - f(t_{j-1})| \right) < +\infty$$

where the supremum is taken over all subdivisions $a = t_0 < t_1 < .... < t_N = b$ of I. Moreover, I can be chosen such that $\mathcal{L}^1(\{x \in I, x + if(x) \in E\}) \geq \frac{9}{10}\mathcal{L}^1(I)$ (since the density of E at M_0 is 1). Now, we claim that there exists a Lipschitz function $h : I \to \mathbb{R}$ such that $\mathcal{L}^1(\{x \in I; h(x) = f(x)\}) \geq \frac{1}{10}\mathcal{L}^1(I)$ and therefore the Lipschitz graph G of g satisfies the required properties. Without a loss of generality, we can assume that $f(a) = f(b)$. Before proving the claim, we recall the well known "Rising Sun Lemma" due to F. Riesz (see for instance [**33**], page 232).

LEMMA 73. *Let ϕ be a continuous function on a closed interval $[c, d]$ such that $\phi(c) = \phi(d)$ and let $\Lambda = \{x \in [c, d];$ there exists $\xi \in [c, x)$ such that $\phi(\xi) < \phi(x)\}$. Then there exist pairwise disjoint open intervals $I_k = (c_k, d_k)$, $k \in K$, such that $\Lambda = \cup_{k\in K} I_k$ and $\phi(c_k) \leq \phi(d_k)$ for any $k \in K$.*

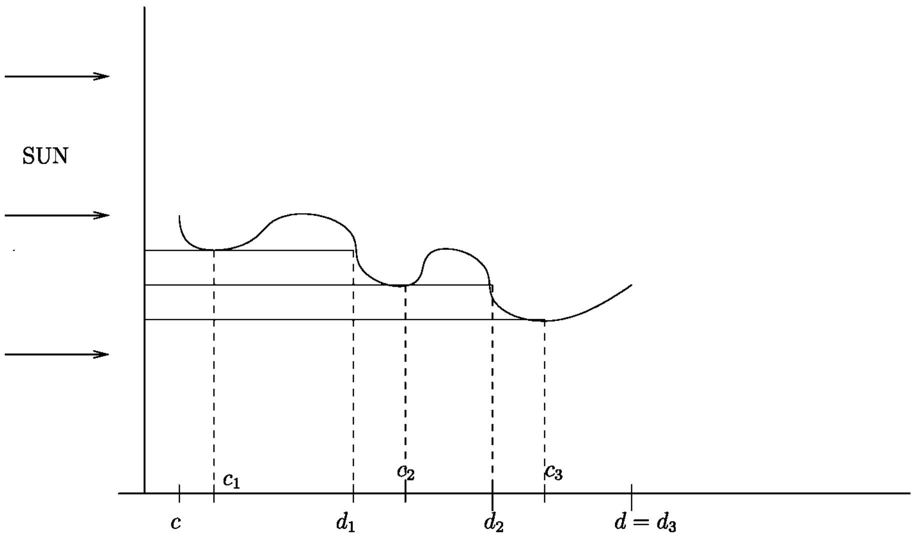

FIGURE 6: The Rising Sun Lemma (in the case $\phi(c) \neq \phi(d)$)

Remark. In fact, $\phi(c_k) = \phi(d_k)$ except perhaps for an index k_0 such that $d = d_{k_0}$.

Set $g(x) = f(x) - \dfrac{4F}{b-a}x$. By applying the rising sun lemma to g, we get pairwise disjoint intervals $I_k = (a_k, b_k)$, $k \in K$, such that

- For any x, $y \in [a,b] \setminus \cup_k I_k$ with $x \leq y$, $g(y) \leq g(x)$, that is

$$f(y) - f(x) \leq \frac{4F}{(b-a)}(y-x). \tag{44}$$

- For any $k \in K$, $f(b_k) - f(a_k) \geq \dfrac{4F}{b-a}(b_k - a_k)$. Thus, by definition of F,

$$\sum_k \mathcal{L}^1(I_k) = \sum_k (b_k - a_k) \leq \frac{b-a}{4F} \sum_k (f(b_k) - f(a_k)) \leq \frac{b-a}{4}.$$

Set

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \notin \cup_k I_k; \\ f(a_j) + \dfrac{4F}{b-a}(x - a_j) & \text{if } x \text{ belongs to some } I_j. \end{cases} \tag{45}$$

Then, $\mathcal{L}^1(\{x \in [a,b], f(x) \neq \tilde{f}(x)\}) \leq \sum_k \mathcal{L}^1(I_k) \leq \dfrac{b-a}{4} = \dfrac{\mathcal{L}^1(I)}{4}$. Moreover, if x, $y \in I$ with $x \leq y$, $\tilde{f}(y) - \tilde{f}(x) \leq \dfrac{4F}{b-a}(y-x)$. The proof of this estimate is divided into 4 cases. Set $\Delta(x,y) = \tilde{f}(y) - \tilde{f}(x)$.

Case 1. x, $y \notin \cup_k I_k$.

Then, $\Delta(x,y) = f(y) - f(x) \leq \dfrac{4F}{b-a}(y-x)$ by (44).

Case 2. $x \in I_j$, $y \in I_{j'}$ for some $j, j' \in K$.

Then, if $j = j'$, $\Delta(x,y) = \frac{4F}{b-a}(y-x)$. Moreover, if $j \neq j'$, then $\Delta(x,y) = g(a_{j'}) - g(a_j) + \frac{4F}{b-a}(y-x) \leq \frac{4F}{b-a}(y-x)$.

Case 3. $y \notin \cup_k I_k$ and $x \in I_j$ for some $j \in K$.

Then, $\Delta(x,y) = f(y) - f(a_j) - \frac{4F}{b-a}(x-a_j) \leq \frac{4F}{b-a}(y-x)$ by (44).

Case 4. $y \in I_j$ for some $j \in K$ and $x \notin \cup_k I_k$. Then, $\Delta(x,y) = f(a_j) - f(x) + \frac{4F}{b-a}(y-a_j) \leq \frac{4F}{b-a}(y-x)$ by (44).

Thus, we get

$$\tilde{f}(y) - \tilde{f}(x) \leq \frac{4F}{b-a}(y-x). \tag{46}$$

Applying now the rising sun lemma to $x \to -\tilde{f}(x) - \frac{4F}{b-a}x$, we get pairwise disjoint open intervals $J_k = (c_k, d_k)$ such that $\tilde{f}(y) - \tilde{f}(x) \geq \frac{-4F}{b-a}(y-x)$ (whenever $x < y$ with x, $y \notin \cup_k J_k$) and $\sum_k \mathcal{L}^1(J_k) \leq \frac{b-a}{4}$. Set

$$h(x) = \begin{cases} \tilde{f}(x) & \text{if } x \notin \cup_k J_k; \\ \tilde{f}(c_j) - \frac{4F}{b-a}(x-c_j) & \text{if } x \in J_j \text{ for some j.} \end{cases} \tag{47}$$

For the same reason as above, $\mathcal{L}^1(\{x \in [a,b]; h(x) \neq \tilde{f}(x)\}) \leq \frac{b-a}{4}$, and hence $\mathcal{L}^1(\{x \in [a,b], h(x) \neq f(x)\}) \leq \frac{b-a}{2}$. Moreover, if x, $y \in [a,b]$ with $x \leq y$,

$$h(y) - h(x) \geq \frac{-4F}{b-a}(y-x). \tag{48}$$

Using (44) and dividing as previously the proof into 4 parts, we also get

$$h(y) - h(x) \leq \frac{4F}{b-a}(y-x). \tag{49}$$

Therefore, by (48) and (49), $|h(x) - h(y)| \leq \frac{4F}{b-a}|y-x|$ and h is Lipschitz with constant $\leq \frac{4F}{b-a}$. □

Remark. In fact, $\frac{F}{b-a}$ can be choosen as small as desired (to see this, use the Lebesgue differentiation theorem). Therefore, we can only consider the case of subsets of Lipschitz graphs with small constant.

From now on, we assume that E is a subset of a Lipschitz graph Γ such that $H^1(E) > 0$. As explained before, we would like to use the strategy described at the end of the previous section. Indeed, the Cauchy operator $\mathcal{C}_\Gamma$ is bounded on $L^2(\Gamma)$, and therefore (by theorem 48) is of weak type $(1,1)$ (see chapter 4). Recall that this means that the truncated Cauchy operator $\mathcal{C}_\Gamma^\varepsilon$ is bounded (uniformly in ε) on $L^2(\Gamma, dH^1_{\restriction E})$ and is of weak type $(1,1)$. Note that, by the previous remark, we

only need the boundedness of the Cauchy operator on Lipschitz graphs with small constant (Calderòn's theorem). The idea to conclude is clear: First, apply theorem 71 to $E \subset \Gamma$ and to $\mathcal{C}_\varepsilon^\Gamma$ to get a sequence of functions (h_ε) whose Cauchy transforms are bounded, then take $\varepsilon \to 0$ to get a (non constant) function h supported on E such that $\int_\Gamma h dH^1 \geq \frac{H^1(E)}{2}$ and $||\frac{1}{z} * (hdH^1_{\restriction E})||_\infty \leq C$. Then, $\gamma(E) > 0$, since the Cauchy transform of a positive measure supported on E is a typical example of an analytic function outside E. But, there are two problems:
(i) $\mathcal{C}_\Gamma^\varepsilon$ does not map $\mathcal{M}(\Gamma)$ to $C_0(\Gamma)$.
(ii) The conclusion gives that $\mathcal{C}_\Gamma^\varepsilon h_\varepsilon$ is bounded on Γ, but not on $\mathbb{C} \setminus \Gamma$.
We now explain how to solve these two problems.
For the first one, consider a test function $\phi \in C_0^\infty(\mathbb{C})$ such that $\phi = 0$ in $B(0, \eta)$ for some small $\eta > 0$ and $\phi(z) = 1$ for $|z| > M$ where $M > 0$ is big. If μ is a Borel measure supported on Γ, set $\mathcal{C}_\phi^\varepsilon \mu(x) = \int_\Gamma (x-y)^{-1} \phi\left(\frac{x-y}{\varepsilon}\right) d\mu(x)$.

Proposition 74. *With the same notations as above,*
(i) $\mathcal{C}_\phi^\varepsilon$ *maps* $\mathcal{M}(\Gamma)$ *to* $C_0(\Gamma)$*;*
(ii) There exists $C > 0$ *such that*

$$\left| \int_{|x-y| \geq \varepsilon} \frac{f(y)}{y-x} dH^1_{\restriction \Gamma}(y) - \int_\Gamma (x-y)^{-1} \phi\left(\frac{x-y}{\varepsilon}\right) dH^1(y)) \right| \leq C M_\Gamma f(x)$$

whenever $\varepsilon > 0$, $x \in \Gamma$, $f \in L^2(\Gamma)$ *and where*

$$M_\Gamma f(x) = \sup_{R>0} \frac{1}{H^1(\Gamma \cap B(x,R))} \int_{\Gamma \cap B(x,R)} |f(y)| dH^1(y);$$

(iii) $\mathcal{C}_\phi^\varepsilon$ *is (uniformly in* ε*) bounded on* $L^2(\Gamma, dH^1_{\restriction E})$ *and is of weak type* $(1,1)$*.*

Note that (iii) follows from (ii) and the L^2 boundedness of the maximal operator on Γ (see theorem 46). Therefore, we can apply theorem 71 to $\mathcal{C}_\phi^\varepsilon$ instead of $\mathcal{C}_\Gamma^\varepsilon$. We leave the proof of proposition 74 to the reader.

To solve the second problem, we prove the following result.

Proposition 75. *Let* Γ *be a Lipschitz graph in* $\mathbb{C}$*. Assume that* $h_\varepsilon \in L^\infty(\Gamma)$ *and that* $\mathcal{C}_\Gamma^\varepsilon h_\varepsilon \in L^\infty(\Gamma)$ *(uniformly in* ε*). Then, there exists* $C > 0$ *(which does not depend on* ε*) such that*

$$||\mathcal{C}_\Gamma^\varepsilon h_\varepsilon||_{L^\infty(\mathbb{C}\setminus\Gamma)} \leq C||h_\varepsilon||_{L^\infty(\Gamma)} + C||\mathcal{C}_\Gamma^\varepsilon h_\varepsilon||_{L^\infty(\Gamma)}.$$

Proof. Throughout all the proof, we forget the ε's.
Fix $z \in \mathbb{C} \setminus \Gamma$. Let $x_0 \in \Gamma$ such that $|z - x_0| = d(z, \Gamma)$ and set $\delta = |z - x_0|$. Decompose now h: $h = h_1 + h_2$ where h_1 is the restriction of h to $B(x_0, 10\delta)$. Then,

$$|\mathcal{C}_\Gamma h_1(z)| = \left| \int_{\Gamma \cap B(x_0, 10\delta)} \frac{h_1(\xi)}{\xi - z} dH^1(\xi) \right| \tag{50}$$

$$\leq ||h_1||_\infty \delta^{-1} H^1(\Gamma \cap B(x_0, 10\delta)) \tag{51}$$

$$\leq C||h||_\infty \text{ since } \Gamma \text{ is Ahlfors regular.} \tag{52}$$

Moreover, if $y \in B(x_0, 5\delta)$, by some easy computations,

$$|\mathcal{C}_\Gamma h_2(z) - \mathcal{C}_\Gamma h_2(y)| \leq C||h||_\infty. \tag{53}$$

Thus, by (50) and (53), for all $y \in B(x_0, 5\delta) \cap \Gamma$,

$$|\mathcal{C}_\Gamma h(z)| \leq C||h||_\infty + |\mathcal{C}_\Gamma h_2(y)| \tag{54}$$
$$\leq C||h||_\infty + ||\mathcal{C}_\Gamma h||_\infty + |\mathcal{C}_\Gamma h_1(y)|. \tag{55}$$

Furthermore,

$$\begin{aligned}\int_{\Gamma\cap B(x_0,5\delta)} |\mathcal{C}_\Gamma h_1(y)| dH^1(y) &\leq H^1(\Gamma\cap B(x_0,5\delta))^{\frac{1}{2}} \left(\int_{\Gamma\cap B(x_0,5\delta)} |\mathcal{C}_\Gamma h_1(y)|^2 dH^1(y)\right)^{\frac{1}{2}} \\ &\leq C\delta^{\frac{1}{2}} ||\mathcal{C}_\Gamma h_1||_{L^2(\Gamma)} \\ &\leq C\delta^{\frac{1}{2}} ||h_1||_{L^2(\Gamma)} \text{ since } \mathcal{C}_\Gamma \text{ is bounded on } L^2(\Gamma) \\ &\leq C\delta ||h||_\infty.\end{aligned}$$

By integrating (54) on $\Gamma \cap B(x_0, 5\delta)$ with respect to H^1 and by using the previous estimate and the Ahlfors-regularity of Γ, we can easily conclude. □

Let Γ be a Lipschitz graph in $\mathbb{C}$ and let E be a subset of Γ with $0 < H^1(E) < +\infty$. By applying theorem 71 to C_ϕ^ε, we get that for every $\varepsilon > 0$, there exists a function h_ε supported on E such that
(i) $0 \leq h_\varepsilon(x) \leq 1$ whenever $x \in \mathbb{C}$;
(ii) $\displaystyle\int_\Gamma h_\varepsilon(x) dH^1(x) \geq \frac{H^1(E)}{2}$;
(iii)$||\mathcal{C}_\phi^\varepsilon h_\varepsilon||_{L^\infty(\Gamma)} \leq C$. But, by applying proposition 74 and then proposition 75, this implies that $||\mathcal{C}_\Gamma^\varepsilon h_\varepsilon||_{L^\infty(\mathbb{C})} \leq C$ (uniformly in ε).

Now, up to some subsequence, (h_ε) converges (with respect to the weak * topology in L^∞) to a function h which satisfies
(i) $h = 0$ outside E and $0 \leq h(x) \leq 1$ whenever $x \in \mathbb{C}$;
(ii) $\displaystyle\int_\Gamma h(x) dH^1(x) \geq \frac{H^1(E)}{2}$;
(iii) $||\mathcal{C}_\Gamma h||_{L^\infty(\mathbb{C}\setminus\Gamma)} \leq C$.
Therefore, $\gamma(E) > 0$ (since $H^1(E) > 0$) and the proof of theorem 69 is complete.

We give an alternative proof of the Denjoy conjecture using the Garabedian formula. We first recall some basic facts on the theory of Hardy spaces (see [**33**], chapter 10 for more details and proofs, or [**44**]). Let Ω be a simply connected domain in $\mathbb{C}$ (with at least two boundary points) and let $p \in [1, +\infty)$. A function f, analytic in Ω, is said to be in the class $H^p(\Omega)$ if the subharmonic function $|f|^p$ has a harmonic majorant in Ω, that is there exists a function F_p, harmonic in Ω, such that $|f(z)|^p \leq F_p(z)$ for all $z \in \Omega$. A function f, analytic in Ω, is said to belong to the class $E^p(\Omega)$ if there exists a sequence of rectifiable (Jordan) curves Γ_1, ..., Γ_n,...... in Ω tending to the boundary of Ω in the sense that eventually Γ_n surrounds every compact subdomain of Ω, such that

$$\int_{\Gamma_n} |f(z)|^p |dz| \leq C < +\infty.$$

The spaces $H^p(\Omega)$ and $E^p(\Omega)$ are two generalizations of the classical notion of Hardy spaces in the unit disc. In fact, in this case, they coincide. More generally, they coincide if the derivative of a conformal mapping ϕ of the unit disc onto the domain Ω satisfies $a \leq |\phi'(z)| \leq b$ whenever $|z| < 1$.
Assume from now on that Ω is a domain bounded by a rectifiable (Jordan) curve.Then,

each function $f \in E^p(\Omega)$ has a non-tangential limit almost everywhere on Γ. Moreover, $\int_\Gamma |f(z)|^p|dz| < +\infty$. Set $||f||_p = \displaystyle\int_\Gamma |f(z)|^p \frac{|dz|}{2\pi}$. Finally, we mention that if $f \in E^1(\Omega)$, then $f(z) = \displaystyle\frac{1}{2i\pi}\int_\Gamma \frac{f(\xi)}{\xi - z} dH^1(\xi)$ whenever $z \in \Omega$.

We now go through the proof of the Denjoy conjecture. For this, consider Γ a Lipschitz graph. Let $f \in L^2(\Gamma, ds)$ (where ds is the arc length on Γ). Set (if this integral exists) $\mathcal{C}f(z) = \displaystyle\frac{1}{2i\pi}\int_\Gamma \frac{f(\xi)}{z-\xi} dH^1(\xi)$. Then, since the Cauchy operator is bounded on $L^2(\Gamma, ds)$, $\mathcal{C}f$ has boundary values $\mathcal{C}^*f$ on Γ and $\mathcal{C}^*f \in L^2(\Gamma, ds)$. This implies that $\mathcal{C}f \in E^2(\Omega)$ (where Ω is the domain of $\mathbb{C}$ whose boundary is Γ). But,

$$\gamma(\Gamma)^{\frac{1}{2}} = \sup\{|h'(\infty)|; h \in E^2(\Omega), ||h||_{E^2(\Omega)} \le 1\}. \tag{56}$$

This have been proved by P. Garabedian [**41**] by introducing the dual extremal problem

$$\inf\left(||g||_{E^1(\Omega)}; g \in E^1(\Omega), g(\infty) = 1\right).$$

Assume that E is a compact subset of Γ and approximate E by a finite union $\tilde{E}$ of subarcs of Γ. Then, by the discussion above, $\mathcal{C}\chi_{\tilde{E}} \in E^2(\Omega)$ since $\chi_{\tilde{E}} \in L^2(\Gamma, ds)$. Moreover, $|(\mathcal{C}\chi_{\tilde{E}})'(\infty)| = \displaystyle\frac{1}{2\pi} H^1(\tilde{E})$.
Thus, if $H^1(E) > 0$, by (56), $\gamma(\tilde{E}) > 0$ and hence $\gamma(E) > 0$. See [**60**] for more details.

Note that the two proofs of the Denjoy conjecture we have presented are based on a duality argument. In fact, no constructive method to finding a non-constant bounded analytic function in $\mathbb{C} \setminus E$ is known (unless Γ is a smooth curve, see [**42**]).

4. Proof of the Vitushkin conjecture

Let $E \subset \mathbb{C}$ be an Ahlfors regular set with dimension 1. Assume that $\gamma(E) > 0$. Our goal is to construct a rectifiable curve $\Gamma \subset \mathbb{C}$ such that $H^1(E \cap \Gamma) > 0$ (proof of theorem 70).

We start with some basic facts in measure theory (see [**65**] for more details).
Let μ be a complex measure. The total variation of μ for a Borel set $B \subset \mathbb{C}$ is

$$|\mu|(B) = \sup\left\{\sum_{i=1}^k |\mu(B_i)|; B_1, ..., B_k \text{ are disjoint Borel sets such that } B = \cup_{i=1}^k B_i\right\}.$$

Note that $|\mu(B)| \le |\mu|(B)$. We define the total variation of μ by $||\mu|| = |\mu|(\mathbb{C})$. Let (μ_j) be a sequence of complex Radon measures with bounded total variations, that is $\sup_j ||\mu_j|| < +\infty$, then there exists a weakly convergent subsequence of (μ_j). This result is an analogue of those for positive measures described in chapter 3.
Let μ be a positive Radon measure, let $A \subset \mathbb{R}^n$ such that $\mu(A) < +\infty$ and let $\mathcal{B}$ be a family of closed balls such that each point of A is the center of arbitrarily small balls of $\mathcal{B}$ (that is $\inf\{r > 0, B(x,r) \in \mathcal{B}\} = 0$ for any $x \in A$). Then, there exists a countable collection of disjoint balls $B_i \in \mathcal{B}$ such that $\mu(A \setminus \cup_i B_i) = 0$ (Vitali covering theorem).

Step 1. Since $\gamma(E) > 0$, there exists a non constant, analytic and bounded function $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $f'(\infty) = 0$ such that $f'(\infty) = \gamma(E)$. The next result says that f is the Cauchy transform of a complex measure.

THEOREM 76. *Let $E \subset \mathbb{C}$ be a compact set with $H^1(E) < +\infty$ and let $f : \mathbb{C} \setminus E \to \mathbb{C}$ be analytic with $||f||_\infty \leq 1$ and $f(\infty) = 0$. Then, there exists a complex Radon measure σ such that Supp$\sigma \subset E$, $|\sigma(B(z,r))| \leq r$ for all $z \in \mathbb{C}$ and $r > 0$, and $f(z) = \displaystyle\int \frac{d\sigma(\xi)}{z-\xi}$ for all $z \in \mathbb{C} \setminus E$.*
Moreover, there exists a Borel function $\phi : E \to \mathbb{C}$ such that $||\phi||_\infty \leq 1$, $\sigma(A) = \int_A \phi(\xi) dH^1(\xi)$ for all Borel set A, and $f(z) = \displaystyle\int_E \frac{\phi(\xi)}{z-\xi} dH^1(\xi)$ for all $z \in \mathbb{C} \setminus E$.

PROOF. The beginning of the proof is similar to those of theorem 64 (A).
Since $H^1(E) < +\infty$, for any $k \in \mathbb{N}^*$, we can cover E with closed discs $B_{k,1}, ..., B_{k,n_k}$ such that
(i) $\mathrm{diam}(B_{k,j}) < \dfrac{1}{k}$ for any $j = 1, ..., n_k$;
(ii) $\sum_{j=1}^{n_k} \mathrm{diam} B_{k,j} \leq 2H^1(E) + \dfrac{1}{k}$.
Set $\Gamma_k = \partial(\cup_{j=1}^{n_k} B_{k,j})$. Then, for any $z \in \mathbb{C} \setminus \cup_{j=1}^{n_k} B_{k,j}$,

$$f(z) = -\frac{1}{2i\pi} \int_{\Gamma_k} \frac{f(\xi)}{\xi - z} dH^1(\xi).$$

Consider now the complex Radon measure σ_k defined by

$$\int \psi(\xi) d\sigma_k(\xi) = -\frac{1}{2i\pi} \int \psi(\xi) f(\xi) dH^1(\xi)$$

for all continuous functions $\psi : \mathbb{C} \to \mathbb{C}$.
Thus, we have $f(z) = \displaystyle\int \frac{1}{\xi - z} d\sigma_k(\xi)$ for all z outside Γ_k and the σ_k's have uniformly bounded total variations:

$$||\sigma_k|| = |\sigma_k|(\mathbb{C}) \leq \frac{1}{2\pi} \int_{\Gamma_k} |f(\xi)| dH^1(\xi) \leq H^1(E) + 1.$$

Hence, we can extract a subsequence converging to a complex measure σ. Of course, supp$\sigma \subset E$ and $f(z) = \displaystyle\int \frac{d\sigma(\xi)}{\xi - z}$ for $z \in \mathbb{C} \setminus E$.
We should now estimate $|\sigma(B(z,r))|$.
Let D be an open disc with

$$H^1(E \cap \partial D) = 0 \text{ and } \int_{\partial D} \int |\xi - z|^{-1} d|\sigma|(\xi) dH^1(z) < +\infty. \tag{57}$$

Note that $-\dfrac{1}{2i\pi} \displaystyle\int_{\partial D} \frac{dH^1(\xi)}{\xi - z} = \chi_D(z)$ Hence, By Fubini's theorem, we get

$$\frac{-1}{2i\pi} \int_{\partial D} f(z) dH^1(z) = \frac{-1}{2i\pi} (\int_{\partial D} \left(\int \frac{d\sigma(\xi)}{\xi - z} \right) dH^1(z) = \int \left(\frac{-1}{2i\pi} \int_{\partial D} \frac{dH^1(z)}{\xi - z} \right) d\sigma(\xi) = \sigma(D).$$

Therefore, $|\sigma(D)| \leq \dfrac{1}{2\pi} H^1(\partial D) = \dfrac{1}{2} \mathrm{diam} D$. We claim that this estimate is true for all discs D of $\mathbb{C}$. To see this, fix now $z_0 \in \mathbb{C}$. Since $H^1(E) < +\infty$, except perhaps for countably many $r > 0$, $H^1(E \cap \partial D(z_0, r)) = 0$.
Moreover, since $\displaystyle\int_{D(z,R)} |\xi - z|^{-1} d\mathcal{L}^2(\xi) < +\infty$ for all $z \in \mathbb{C}$, all $R > 0$, $\int |\xi - z|^{-1} d|\sigma|(\xi) < +\infty$ and thus, by Fubini's theorem, $\int_{\partial D(z_0,R)} \int |\xi - z|^{-1} d|\sigma|(\xi) dH^1(z) < +\infty$ for all $R > 0$. Therefore, for almost every $R > 0$, (57) holds for $D = D(z_0, R)$

and hence $|\sigma(D(z_0,R))| \leq R$. Fix now $R > 0$ and let $\varepsilon > 0$. Then, there exists $R_\varepsilon \in [R, R+\varepsilon]$ such that $|\sigma(D(z_0,R_\varepsilon))| \leq R_\varepsilon$. Thus, $|\sigma(D(z_0,R))| \leq |\sigma(D(z_0,R_\varepsilon)| \leq R_\varepsilon \leq R+\varepsilon$. By taking $\varepsilon \to 0$, we get $|\sigma(D(z_0,R))| \leq R$ and the claim is proved. Assume now that $A \subset \mathbb{C}$ is a Borel set with $H^1(A) = 0$ and let $\varepsilon > 0$. Then, we can cover A by discs D_i such that $\sum_i \mathrm{diam} D_i \leq 2\varepsilon$. Therefore,

$$|\sigma(A)| \leq \sum_i |\sigma(D_i)| \leq \frac{1}{2}\sum_i \mathrm{diam} D_i \leq \varepsilon \tag{58}$$

By taking $\varepsilon \to 0$, this yields

$$|\sigma(A)| = 0. \tag{59}$$

Now we claim that, for general Borel sets $A \subset E$, $|\sigma|(A) \leq H^1(A)$. Note that by definition of the total variation, it is enough to prove that $|\sigma(A)| \leq H^1(A)$ for all Borel sets $A \subset \mathbb{R}^n$. To prove this, fix $\varepsilon > 0$. Denote by $\mathcal{B}$ the collection of discs $D(x,r)$ centered on A such that $(1+\varepsilon)H^1(A \cap D(x,r)) \geq r$. By proposition 9, if we set $A' = \{x \in A, \theta^{*1}(A,x) \geq \frac{1}{2}\}$, then $H^1(A \setminus A') = 0$ and by (59), $|\sigma(A \setminus A')| = 0$. Moreover, for every $x \in A'$, $\inf\{r; D(x,r) \in \mathcal{B}\} = 0$. Therefore, by applying Vitali's covering theorem, we find disjoint discs $D_i = D(x_i, r_i)$ such that $(1+\varepsilon)H^1(A \cap D_i) \geq r_i$, $|\sigma|(A' \setminus \cup_i D_i) = 0$. Then,

$$\begin{aligned}
|\sigma(A)| &\leq |\sigma(A')| + |\sigma(A \setminus A')| \\
&\leq |\sigma(A')| \\
&\leq |\sigma(\cup_i D_i)| + |\sigma(A' \setminus \cup_i D_i)| \\
&\leq \sum_i |\sigma(D_i)| \\
&\leq \sum_i r_i \\
&\leq (1+\varepsilon)\sum_i H^1(A \cap D_i) \\
&\leq (1+\varepsilon)H^1(A).
\end{aligned}$$

This implies that $|\sigma(A)| \leq H^1(A)$, and the claim is proved. It follows that σ is absolutely continuous with respect to $H^1_{\lceil E}$ with Radon-Nykodyn derivative ϕ such that $||\phi||_\infty \leq 1$. This finishes the proof (which is taken from [**65**]).

Step 2. Starting with the function ϕ (given by theorem 74) and using a stopping time argument, we construct an Ahlfors regular subset F of E and a para-accretive function b supported on F such that $H^1(E \cap F) > 0$ and $\mathcal{C}_F(b) \in BMO(H^1_{\lceil F})$. Thus, by the $T(b)$ theorem, the Cauchy operator $\mathcal{C}_F$ is bounded on $L^2(H^1_{\lceil F})$.

THEOREM 77. [**18**] *Let $E \subset \mathbb{C}$ be an Ahlfors regular set such that $\gamma(E) > 0$. Then there exists an Ahlfors regular set $F \subset E$ such that the Cauchy operator $\mathcal{C}_F$ is bounded on $L^2(H^1_{\lceil F})$ and such that $H^1(E \cap F) > 0$.*

Remark. This result has been extended to the case of sets supporting a doubling measure by Y. Lin [**58**].

PROOF. By theorem 76, there exists a Borel function $\phi : E \to \mathbb{C}$ such that $||\phi||_\infty \le 1$, $\left|\int_E \frac{\phi(\xi)}{z-\xi} dH^1(\xi)\right| \le 1$ whenever $z \notin E$ and $\int_E \phi(\xi)dH^1(\xi) \ne 0$. Let $\Delta(E)$ be the family of "dyadic cubes" associated to E (see proposition 24). Recall that "dyadic cubes" of $\Delta(E)$ have "small boundary". Let $\varepsilon > 0$. Consider

$$\mathcal{G} = \{Q \in \Delta(E); |\int_Q \phi(\xi)dH^1(\xi)| \ge \varepsilon H^1(Q)\},$$

$$\mathcal{B} = \{Q \in \Delta(E); Q \notin \mathcal{G}, \text{ but the father } R \text{ of } Q \text{ is in } \mathcal{G}\}.$$

Recall that $R \in \Delta(E)$ is the father of $Q \in \Delta_j(E)$ if $R \in \Delta_{j-1}(E)$ and $Q \subset R$, that is R is the "dyadic cube" of the previous generation containing Q. Let $Q \in \mathcal{G}$. Then, there exists $c_Q \in E$ such that $E \cap B(c_Q, 2\delta\text{diam}Q) \subset Q$. Moreover, $Q \subset E \cap B(c_Q, C\text{diam}Q)$ (see proposition 24). Let S_Q be the union of two circles with center c_Q and with radii $\delta\text{diam}Q$ and $\frac{\delta\text{diam}Q}{2}$ respectively. Define g_Q on S_Q by $g_Q = C_1(Q)$ on $\partial D\left(c_Q, \delta\text{diam}Q\right)$ and $g_Q = C_2(Q)$ on $\partial D\left(c_Q, \frac{\delta\text{diam}Q}{2}\right)$ such that $|C_1(Q)|$ and $|C_2(Q)|$ are uniformly bounded, and $\int_{S_Q} g_Q dH^1 = \int_Q \phi dH^1$.

Define $F = (E \setminus \cup_{Q\in\mathcal{B}} Q) \cup (\cup_{Q\in\mathcal{B}} S_Q)$. Let b be the function defined by $b = \phi$ on $E \cap F$ and $b = g_Q$ on any $Q \in \mathcal{B}$.

Then,

$$\begin{aligned}\left|\int_E \phi(\xi)dH^1(\xi) - \int_{E\cap F} \phi(\xi)dH^1(\xi)\right| &\le \sum_{Q\in\mathcal{B}} \left|\int_Q \phi(\xi)dH^1(\xi)\right| \\ &\le \varepsilon \sum_{Q\in\mathcal{B}} H^1(Q) \\ &\le \varepsilon H^1(E)\end{aligned}$$

Thus, since $\int_E \phi(\xi)dH^1(\xi) \ne 0$ and $H^1(E) < +\infty$, $\int_{E\cap F} \phi(\xi)dH^1(\xi) \ne 0$ if $\varepsilon > 0$ is small enough. This implies that $H^1(E \cap F) > 0$.

We claim that F is Ahlfors regular. To see this, first note that there exists $C \ge 1$ such that

$$C^{-1}H^1(S_Q) \le H^1(Q) \le CH^1(S_Q) \tag{60}$$

whenever $Q \in \mathcal{B}$. Furthermore, recall that, by construction, there exists $\delta > 0$ such that, for any $Q \in \mathcal{B}$, any $x \in E^*$,

$$d(x, S_Q) \ge \delta\text{diam}Q \tag{61}$$

where $E^* = E \setminus (\cup_{Q\in\mathcal{B}} Q)$. Consider now $x \in F$ and $R \in (0, \text{diam}F)$.
Case 1. $x \in E^*$.
Write $H^1(F \cap B(x,R)) = T_1(x,R) + T_2(x,R) + T_3(x,R)$ where

$T_1(x,R) = H^1(E^* \cap B(x,R))$,

$T_2(x,R) = \sum_{Q\in\mathcal{B}_1(x,R)} H^1(S_Q \cap B(x,R))$ where $\mathcal{B}_1(x,R)$ is the set of cubes $Q \in \mathcal{B}$ such that $S_Q \cap B(x,R) \ne \emptyset$ and $\text{diam}Q \le 10\delta^{-1}R$,

$T_3(x,R) = \sum_{Q\in\mathcal{B}_2(x,R)} H^1(S_Q \cap B(x,R))$ where $\mathcal{B}_2(x,R)$ is the set of cubes $Q \in \mathcal{B}$ such that $S_Q \cap B(x,R) \neq \emptyset$ and $\mathrm{diam}Q > 10\delta^{-1}R$.

Then, since E is Ahlfors-regular,

$$T_1(x,R) \leq H^1(E \cap B(x,R)) \leq CR. \tag{62}$$

Let $Q \in \mathcal{B}_2(x,R)$. Then, Q, $S_Q \subset B(x,CR)$ for some $C>0$ depending on δ. Thus, by (60),

$$T_2(x,R) \leq \sum_{Q\subset B(x,CR)} H^1(S_Q) \leq C \sum_{Q\subset B(x,CR)} H^1(Q) \leq CH^1(E\cap B(x,CR)).$$

Hence,

$$T_2(x,R) \leq CR. \tag{63}$$

Assume now that $\mathcal{B}_2(x,R) \neq \emptyset$. Then, there exists $Q \in \mathcal{B}$ such that $d(x,S_Q) \leq R$ and $\mathrm{diam}Q > 10\delta^{-1}R$. Therefore, by (61), $\delta\mathrm{diam}Q \leq d(x,S_Q) \leq R \leq \frac{\delta}{10}\mathrm{diam}Q$. This estimate is impossible. Hence, $T_3(x,R)=0$ and by (62) and by (63), we get

$$H^1(F \cap B(x,R)) \leq CR. \tag{64}$$

To prove the converse, choose $S \in \Delta(E)$ such that $x \in S$ and $\frac{R}{M_1} \leq \mathrm{diam}S \leq \frac{R}{M_2}$ where $M_1 \geq M_2 > 0$ are big enough. Then,

$$\begin{aligned}
C^{-1}\frac{R}{M_1} &\leq C^{-1}\mathrm{diam}S \qquad (65)\\
&\leq H^1(E\cap B(c_S,\delta\mathrm{diam}S)) \qquad (66)\\
&\leq H^1(E^*\cap B(c_S,\delta\mathrm{diam}S)) + \sum_{Q\in\mathcal{B}} H^1(Q\cap B(c_S,\delta\mathrm{diam}S)) \qquad (67)\\
&\leq H^1(E^*\cap B(c_S,\mathrm{diam}S)) + \sum_{Q\in\mathcal{B}} H^1(Q\cap B(c_S,\delta\mathrm{diam}S)). \qquad (68)
\end{aligned}$$

Let $Q \in \mathcal{B}$ such that $Q \cap B(c_S,\delta\mathrm{diam}S) \neq \emptyset$. Then,

$$\delta\mathrm{diam}Q \leq d(x,Q) \leq |x-c_S| + d(c_S,Q) \leq C\mathrm{diam}S + \delta\mathrm{diam}S.$$

Hence, $\mathrm{diam}Q \leq \frac{C}{M_2}R$. Moreover, if M_2 is big enough, for any $y \in S_Q$,

$$|x-y| \leq |x-c_S| + |c_S-y| \leq \frac{C}{M_2}R + C\mathrm{diam}Q + \frac{\delta}{M_2}R < R.$$

Therefore, $S_Q \subset B(x,R)$ and

$$\sum_{Q\in\mathcal{B}} H^1(Q\cap B(C_S,\delta\mathrm{diam}S)) \leq \sum_{S_Q\subset B(x,R)} H^1(Q) \leq C\sum_{S_Q\subset B(x,R)} H^1(S_Q).$$

Finally, from (65), we get

$$\begin{aligned}
C^{-1}R &\leq CH^1(E^*\cap B(x,R)) + C\sum_{S_Q\subset B(x,R)} H^1(S_Q)\\
&\leq CH^1(F\cap B(x,R))
\end{aligned}$$

Thus,

$$H^1(F\cap B(x,R)) \geq C^{-1}R. \tag{69}$$

Case 2. $x \in S_Q$ for some $Q \in \mathcal{B}$.
Recall that there exists $c_Q \in E$ such that $|x - c_Q| \leq \delta \text{diam} Q$. Moreover, $d(x, E^* \setminus S_Q) \geq \delta \text{diam} Q$. Assume first that $R < \delta \text{diam} Q$. Then, $H^1(F \cap B(x,R)) = H^1(S_Q \cap B(x,R))$. Since S_Q is Ahlfors regular, we get

$$C^{-1}R \leq H^1(F \cap B(x,R)) \leq CR.$$

From now on, suppose that $R \geq \delta \text{diam} Q$. Then, $B(x,R) \subset B(c_Q, 10R)$ and by adapting the method using to prove (64), we get $H^1(F \cap B(c_Q, 10R)) \leq CR$ and thus $H^1(F \cap B(x,R)) \leq CR$.
Conversely, assume first that $\delta \text{diam} Q \leq R \leq \text{diam} Q$. Hence, since S_Q is Ahlfors-regular,

$$H^1(F \cap B(x,R)) \geq H^1(S_Q \cap B(x,R)) \geq C^{-1}R.$$

If $R > \text{diam} Q$, $B\left(c_Q, \dfrac{R}{10}\right) \subset B(x,R)$. By adapting the method using to prove (69) in case 1, we get $H^1(F \cap B(c_Q, \frac{R}{10})) \geq C^{-1}R$. Thus, $H^1(F \cap B(x,R)) \geq C^{-1}R$.

We now equip F with a dyadic decomposition by "cubes". For this, consider all cubes $Q \in \Delta(E) \setminus \mathcal{B}$, all S_Q, $Q \in \mathcal{B}$ together with each two circles of S_Q together with subsets of these circles obtaining by bisecting each circles into 2 semi-circles, then repeatedly bisecting the semi-circles and resulting arcs. This family is a family of "dyadic cubes" of F. We will denote it by $\Delta(F)$. Note that, by construction, "dyadic cubes" of $\Delta(F)$ satisfy also the "small boundary" condition.

To prove that $\mathcal{C}_F$ is bounded on $L^2(H^1_{\lceil F})$, we check that the hypothesis of the Tb theorem (theorem 50) are satisfied.

First, we prove that b is special para-accretive (with respect to $\Delta(F)$) that is

$$\left| \int_Q b(\xi) dH^1(\xi) \right| \geq C^{-1} H^1(Q) \text{ for any } Q \in \Delta(F).$$

Indeed, if $Q \in \Delta(E) \cap \Delta(F)$, by construction,

$$\left| \int_Q b(\xi) dH^1(\xi) \right| = \left| \int_Q \phi(\xi) dH^1(\xi) \right| \geq \varepsilon H^1(Q).$$

Moreover, if $Q \in \Delta(F) \setminus (\Delta(E) \cap \Delta(F))$, then Q is a subset of S_R for some $R \in \mathcal{B}$. Thus, since $C_1(R)$ and $C_2(R)$ are uniformly bounded,

$$\left| \int_Q b(\xi) dH^1(\xi) \right| \geq C^{-1} H^1(Q)$$

for some absolute constant $C > 0$.

We now check that, for any $\varepsilon > 0$, $\mathcal{C}_F^\varepsilon b$ belongs to $BMO(H^1_{\lceil F})$ uniformly in ε, where as usual $C_F^\varepsilon b(z) = \displaystyle\int_F \frac{b(\xi)}{z - \xi} dH^1(\xi)$. By some standard arguments (see the discussion in chapter 4), it suffices to prove that for any $Q \in \Delta(F)$,

$$\int_Q |\mathcal{C}_F^\varepsilon (b \chi_Q)| dH^1 \leq C H^1(Q) \tag{70}$$

where $C > 0$ is an absolute constant. Recall that any $Q \in \Delta(E)$ has a "small boundary", that if for $0 \le \tau < 1$,

$$H^1(\{z \in Q; d(z, E \setminus Q) \le \tau \text{diam} Q\}) \le C\tau^{\frac{1}{C}} H^1(Q). \tag{71}$$

Fix $Q \in \Delta(F)$ and $\varepsilon > 0$. We divide the proof of (70) into three cases.

Case 1. Q is a subset of some S_R (which is the union of two concentric circles). Using the fact that $b\chi_Q$ takes at most two values which are uniformly bounded, you can easily prove that (70) holds in this case.

Case 2. $Q \in \Delta(E)$. Hence, $\mathcal{C}_F^\varepsilon(b\chi_Q) = \mathcal{C}_E^\varepsilon(\phi\chi_Q)$. First, we claim that, for any $z \in \mathbb{C}$,

$$|\mathcal{C}_E^\varepsilon(\phi)(z)| \le C \tag{72}$$

where $C > 0$ is an absolute constant which does not depend on ε.

Since $z \to \displaystyle\int_E \frac{\phi(\xi)}{z-\xi} dH^1(\xi)$ is bounded by 1 outside E, it is enough to consider the case $z \in E$. Fix $z_0 \in E$ and choose positive constants $\varepsilon < \delta'' < \delta' < \delta$ such that

(a) $\delta'' \le 2\varepsilon$,

(b) $\delta' \le \dfrac{\delta^2}{1+\delta}$.

Then, if $z \notin E$,

$$\left|\int_{E \setminus D(z_0,\varepsilon)} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi)\right| \le I(z) + \left|\int_E \frac{\phi(\xi)}{\xi - z} dH^1(\xi)\right| \tag{73}$$

$$\le I(z) + 1 \tag{74}$$

where

$$I(z) = \left|\int_{E \setminus D(z_0,\varepsilon)} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi) - \int_E \frac{\phi(\xi)}{\xi - z} dH^1(\xi)\right|.$$

Therefore, our strategy is to choose $z \notin E$ such that $I(z) \le C$ for some absolute constant $C > 0$. Write $I(z) \le I_1(z) + I_2(z) + I_3(z)$ where

$$\begin{aligned}
I_1(z) &= \left|\int_{E \setminus D(z_0,\delta)} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi) - \int_{E \setminus D(z_0,\delta)} \frac{\phi(\xi)}{\xi - z} dH^1(\xi)\right| \\
I_2(z) &= \left|\int_{D(z_0,\delta) \setminus D(z_0,\varepsilon)} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi) - \int_{D(z_0,\delta) \setminus D(z_0,\delta'')} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi)\right| \\
I_3(z) &= \left|\int_{D(z_0,\delta)} \frac{\phi(\xi)}{\xi - z} dH^1(\xi) - \int_{D(z_0,\delta) \setminus D(z_0,\delta'')} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi)\right|.
\end{aligned}$$

Assume first that $z \in D(z_0, \delta')$. Then,

$$\begin{aligned}
I_1(z) &\le ||\phi||_\infty \int_{E \setminus D(z_0,\delta)} \left|\frac{z - z_0}{(\xi - z)(\xi - z_0)}\right| dH^1(\xi) \\
&\le \frac{\delta'}{\delta(\delta - \delta')} H^1(E) \\
&\le C \text{ by (b).}
\end{aligned}$$

Furthermore,

$$\begin{aligned} I_2(z) &\leq \left| \int_{D(z_0,\delta'')\setminus D(z_0,\varepsilon)} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi) \right| \\ &\leq C||\phi||_\infty \frac{\delta''}{\varepsilon} \text{ since } E \text{ is Ahlfors-regular} \\ &\leq C \text{ by (a).} \end{aligned}$$

Finally, since E is Ahlfors-regular, we can choose $z \in D(z_0, \delta')$ such that

(c) $\dfrac{\delta''}{4} \leq |z_0 - z| \leq \dfrac{\delta''}{2}$

(d) $E \cap D\left(z, \dfrac{\delta''}{10}\right) = \emptyset$.

Then, using (c), (d), and the regularity of E, we get

$$\begin{aligned} I_3(z) &\leq \left| \int_{D(z_0,\delta)\setminus D(z_0,\delta'')} \frac{\phi(\xi)}{\xi - z} dH^1(\xi) - \int_{D(z_0,\delta)\setminus D(z_0,\delta'')} \frac{\phi(\xi)}{\xi - z_0} dH^1(\xi) \right| \\ &+ \left| \int_{D(z_0,\delta'')} \frac{\phi(\xi)}{\xi - z} dH^1(\xi) \right| \\ &\leq C||\phi||_\infty \int_{E\cap(D(z_0,\delta)\setminus D(z_0,\delta''))} \frac{\delta''}{|\xi - z|^2} dH^1(\xi) + C||\phi||_\infty \\ &\leq C. \end{aligned}$$

Remark. To estimate the last integral above, write

$$\int_{E\cap(D(z_0,\delta)\setminus D(z_0,\delta))} \frac{1}{|\xi - z|^2} dH^1(\xi) \leq \sum_{j=1}^{N} \int_{E_j(z)} \frac{1}{|\xi - z|^2} dH^1(\xi)$$

where $E_j(z) = \{\xi \in E, 2^j\delta'' \leq |\xi - z| \leq 2^{j+1}\delta''\}$ and $N \in \mathbb{N}$ satisfies $2^N\delta'' \leq \delta \leq 2^{N+1}\delta''$.

Note that a slight modification of this proof gives, for any ball $B = B(c_B, C\text{diam}Q)$ centered on E such that $Q \subset B\left(c_B, \dfrac{C}{10}\text{diam}Q\right)$,

$$||\mathcal{C}_E^\varepsilon(\phi\varphi_B)||_{L_\infty(\mathbb{C})} \leq C \tag{75}$$

where φ_B is a smooth function supported on $B(c_B, \frac{C}{10}\text{diam}Q)$ such that $\varphi_B = 1$ on Q. Choose now a ball B_Q centered on E and a smooth function φ_{B_Q} supported on B_Q as above. Then, by (75),

$$\int_Q \left|\mathcal{C}_E^\varepsilon(\phi\chi_Q)dH^1\right| \leq \int_Q \left|\mathcal{C}_E^\varepsilon(\phi\varphi_{B_Q})\right| dH^1 + \int_Q \left|\mathcal{C}_E^\varepsilon(\phi(\varphi_{B_Q} - \chi_Q))\right| dH^1 \tag{76}$$

$$\leq CH^1(Q) + \int_Q \left|\mathcal{C}_E^\varepsilon(\phi(\varphi_{B_Q} - \chi_Q))\right| dH^1. \tag{77}$$

Consider now a bounded function g supported on E such that supp $g \subset B_Q \setminus Q$. Then,

$$\int_Q |\mathcal{C}_E^\varepsilon(g)| dH^1 \leq CH^1(Q). \tag{78}$$

To see this, write

$$\int_Q |\mathcal{C}_E^\varepsilon(g)|dH^1 = \int_Q \left| \int_{E\setminus D(z,\varepsilon)} \frac{g(\xi)}{\xi - z} dH^1(\xi) \right| dH^1(z).$$

Since g is supported outside Q, we can omit ε. Set $E_j(z) = \{\xi \in (E \cap B_Q) \setminus Q; 2^{-j}\text{diam}Q \le d(z,\xi) \le 2^{-j+1}\text{diam}Q\}$ and $F(z) = \{\xi \in (E \cap B_Q) \setminus Q; d(z,\xi) \ge \text{diam}Q\}$. Note that, if $E_j(z) \neq \emptyset$, then $d(z, E \setminus Q) \le 2^{-j+1}\text{diam}Q$. Thus, using the fact that Q has a small boundary (see (71)) and the regularity of E, we get

$$\begin{aligned}
\int_Q |\mathcal{C}_E(g)|dH^1 &\le \int_Q \int_{F(z)} \left| \frac{g(\xi)}{\xi - z} \right| dH^1(\xi) dH^1(z) \\
&+ \sum_{j=1}^{\infty} \int_Q \left| \int_{E_j(z)} \frac{g(\xi)}{\xi - z} dH^1(\xi) \right| dH^1(z) \\
&\le CH^1(Q) + \sum_{j=1}^{+\infty} (2^{-j})^{\frac{1}{C}} H^1(Q) \\
&\le CH^1(Q).
\end{aligned}$$

Therefore, (77) and (78) give (70).

Note that, since "dyadic cubes" of $\Delta(F)$ satisfy also the "small boundary" condition, we can adapt the previous proof to get, for any $Q \in \Delta(F)$,

$$\int_Q |\mathcal{C}_F^\varepsilon(\phi\chi_Q)|dH^1 \le CH^1(Q). \tag{79}$$

Case 3. $Q = (Q \cap E) \cup (\cup_{R\in\Lambda(Q)} S_R)$. Write $\mathcal{C}_F^\varepsilon(b\chi_Q) = \mathcal{C}_F^\varepsilon(\phi\chi_Q) + \sum_{R\in\Lambda(Q)} \mathcal{C}_F^\varepsilon((b - \phi)\chi_{S_R})$. Then, by the result of case 1 and (79), this yields

$$\begin{aligned}
\int_Q |\mathcal{C}_F^\varepsilon(b\chi_Q)|dH^1 &\le \int_Q |\mathcal{C}_F^\varepsilon(\phi\chi_Q)|dH^1 + \sum_{R\in\Lambda(Q)} \int_{Q\setminus S_R} |\mathcal{C}_F^\varepsilon((b-\phi)\chi_{S_R})|dH^1 \\
&+ \sum_{R\in\Lambda(Q)} \int_{S_R} |\mathcal{C}_F^\varepsilon(b\chi_{S_R})|dH^1 + \sum_{R\in\Lambda(Q)} \int_{S_R} |\mathcal{C}_F^\varepsilon(\phi\chi_{S_R})|dH^1 \\
&\le CH^1(Q) + \sum_{R\in\Lambda(Q)} \int_{Q\setminus S_R} |\mathcal{C}_F^\varepsilon((b-\phi)\chi_{S_R})|dH^1 \\
&+ C \sum_{R\in\Lambda(Q)} H^1(S_R) + C \sum_{R\in\Lambda(Q)} H^1(S_R) \\
&\le CH^1(Q) + \sum_{R\in\Lambda(Q)} \int_{Q\setminus S_R} |\mathcal{C}_F^\varepsilon((b-\phi)\chi_{S_R})|dH^1.
\end{aligned}$$

Now, we claim that

$$\int_{Q\setminus S_R} |\mathcal{C}_F^\varepsilon((b-\phi)\chi_{S_R})|dH^1 \le CH^1(S_R).$$

Therefore, $\int_Q |\mathcal{C}_F^\varepsilon(b\chi_Q)|dH^1 \le CH^1(Q)$ and the proof of (70) is complete.
To prove the claim, set $g = (b - \phi)\chi_{S_R}$. Then, g is supported on S_R, $g \in L^\infty(\mathbb{C})$ and (by construction of b) $\int_{\mathbb{C}} g dH^1 = 0$. Let ξ_R be the center of S_R and set for $z \in Q \setminus S_R$,

$\Phi_z(\xi) = \dfrac{1}{\xi - z}$. Then, for ξ, $\xi' \in S_R$, $|\Phi_z(\xi) - \Phi_z(\xi')| \leq C\dfrac{|\xi - \xi'|}{|z - \xi_R|^2}$. In other words, Φ_z is Lipschitz on S_R with constant less than $\dfrac{C}{|z - \xi_R|^2}$. Hence,

$$\begin{aligned}\left|\int_{S_R} \frac{g(\xi)}{\xi - z} dH^1(\xi)\right| &= \left|\int_{S_R} g(\xi)(\Phi_z(\xi) - \Phi_z(\xi_R)) dH^1(\xi)\right| \\ &\leq \frac{C}{|z - \xi_R|^2} \,\text{diam}R \,||g||_{L^1(S_R)}.\end{aligned}$$

Set $F_j = \{z \in Q \setminus S_R, 2^j \text{diam} S_R \leq |z - \xi_R| \leq 2^{j+1}\text{diam} S_R\}$. Note that $H^1(F_j) \leq C2^j \text{diam}R$. To see this, use the Ahlfors-regularity of F and the fact that $F_j \subset B(y, 10.\ 2^j \text{diam} S_R)$ for some $y \in S_R$.

Then,

$$\begin{aligned}\int_{Q\setminus S_R} \left|\int_{S_R} \frac{g(\xi)}{\xi - z} dH^1(\xi)\right| dH^1(z) &\leq C \sum_{j=1}^{+\infty} \int_{F_j} \left|\int_{S_R} \frac{g(\xi)}{\xi - z} dH^1(z)\right| dH^1(z) \\ &\leq C||g||_{L^1(S_R} \\ &\leq CH^1(S_R)\end{aligned}$$

This type of estimate is classical in the theory of singular integral operators (see the discussion in [**17**], pages 17-18).

We can now conclude that, by the Tb theorem, the Cauchy operator is bounded on $L^2(F, dH^1_{\restriction F})$. □

Step 3. Since F is Ahlfors-regular and C_F is bounded on $L^2(F, dH^1_{\restriction F})$, F is uniformly rectifiable (by theorem 52). Therefore, there exists a rectifiable curve $\Gamma \subset \mathbb{C}$ such that $H^1(E \cap \Gamma) \geq H^1(F \cap \Gamma) > 0$. The proof of the Vitushkin conjecture in the Ahlfors-regular case is now complete. □

We now explain how to modify the proof in the general case. The fist step remains unchanged, since the only property of E we need in theorem 76 is $H^1(E) < +\infty$. The first problem in step 2 was to construct a "dyadic" decomposition of the set E. This has been done in [**27**]. Using this and the measure $f dH^1_{\restriction E}$ given by the theorem 76, G. David and P. Mattila adapt M. Christ's stopping time argument to construct a new finite measure $g d\nu$ such that

- ν is a positive measure with linear growth.
- $g : \mathbb{C} \to \mathbb{C}$ is bounded and accretive, that is $\text{Re} g(x) \geq C$ for any $x \in \mathbb{C}$.
- $\int f d\mu = \int g d\nu = m > 0$.
- There exists a Borel set $F \subset E$ such that $C^{-1}\nu(A) \leq \mu(A) \leq C\nu(A)$ for all Borel subsets $A \subset F$ and $\nu(F) \geq \dfrac{m}{2}$.
- The Cauchy transform of $g d\nu$ is in a BMO type space.

The second problem was the lack of a Tb theorem in the setting of nonhomogeneous spaces (that is spaces equipped with a non doubling measure). In [**26**], G. David proved a version of the Tb theorem in this general situation and used it to complete the proof of the Vitushkin conjecture (see [**78**] for an other proof following the same scheme). Indeed, by applying the Tb theorem with $b = g$ in the space $\text{Supp}\mu$, we get that the truncated Cauchy operator $\mathcal{C}^\varepsilon_\nu$ is bounded on $L^2(\nu)$ uniformly in ε. In particular, $||\mathcal{C}^\varepsilon_\nu(1)||_{L^2(\nu)} \leq C$ where $C > 0$ does not depend on ε. From this, by

adapting the arguments given in the section 7 of the chapter 4, we get $c^2(\nu) < +\infty$. Thus, by theorem 28 (or more precisely a slight modification of it), Suppν is rectifiable. Therefore, $F \cap \text{Supp}\nu$ is also rectifiable. Hence, there exists a rectifiable curve Γ such that $H^1(\Gamma \cap F) > 0$ and the proof is complete.
It should be noted that other versions of the Tb theorems are given in [**100**] and [**77**] where harmonic analysis in the setting of nonhomogeneous spaces was developed. As wrote J. Verdera in his nice survey about the subject [**107**], "This has come as a great surprise to those, the author among them, that felt that homogeneous spaces were not only a convenient setting for developing Calderòn-Zygmund theory, but that they were essentially the right context" ! We conclude by giving a version of the $T1$ theorem for the Cauchy integral operator (similar to those given in [**100**], [**77**] and [**105**]).

Theorem 78. *Let μ be a positive Radon measure in $\mathbb{C}$ with linear growth. Assume that there exists $C > 0$ such that*

$$\int_D |\mathcal{C}_\mu^\varepsilon(\chi_D)|^2 d\mu \leq C\mu(D) \tag{80}$$

for any disc D in $\mathbb{C}$ and any $\varepsilon > 0$. Then, the Cauchy operator $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$.

The conclusion of the theorem says that there exists $C > 0$ such that $\displaystyle\int |\mathcal{C}_\mu^\varepsilon(f)|^2 d\mu \leq C \int |f|^2 d\mu$ whenever $f \in L^2(\mu)$ and $\varepsilon > 0$ and where

$$\mathcal{C}_\mu^\varepsilon(f)(z) = \int_{\xi \in \mathbb{C}\setminus D(z,\varepsilon)} \frac{f(\xi)}{\xi - z} d\mu(\xi).$$

We have seen in chapter 4 that, in the doubling case, the condition (80) is equivalent to $\mathcal{C}_\mu(1) \in BMO(\mu)$. Note that by the proof given in [**105**], you can replace (80) by

$$\int_D \int_D \int_D c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z) \leq C\text{diam}D \tag{81}$$

for any disc D in $\mathbb{C}$.

5. The Vitushkin conjecture for sets with infinite length

We now turn back to the original Vitushkin conjecture. Recall that it stated that the removability of a compact set of $\mathbb{C}$ is equivalent to the fact that the Favard length of the set is 0 (without assuming that the set has finite length). As we said before, it is not true, but the story of this is quite amazing. First, P. Mattila [**63**] showed that one of this condition ($\gamma(E) = 0$) is conformally invariant, whereas the other condition is not. However, his method does not indicate which implication is false. Later, P. Jones and T. Murai [**51**] gave an example of set (of infinite length !) with zero Favard length, but non zero analytic capacity. In this section, we present a more simple example due to H. Joyce and P. Mörters [**53**]. Note that it is still open to determine if a set with non zero Favard length is removable or not for bounded analytic functions (see the discussion in the last chapter). Recall that, if $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function with $h(0) = 0$, then Λ_h denotes the Hausdorff measure associated to the gauge h (see section 3 of the first chapter).

THEOREM 79. *There exists a compact set $E \subset \mathbb{C}$ such that*
(i) the projection of E in any direction has zero length;
(ii) $0 < \Lambda_h(E) < +\infty$ for a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ fulfilling (for some $\varepsilon > 0$) $\int_0^\varepsilon h(r)^\alpha r^{-\alpha-1} dr < +\infty$ for every $\alpha > 1$.

By theorem 42, this implies that the Menger curvature of the restriction of Λ_h to E has finite curvature. Moreover, there exists a subset $F \subset E$ such that the restriction of Λ_h to F has linear growth. To see this, fix $\delta \in (0, \varepsilon)$ small enough. We first claim that there exists a compact subset F of E such that $0 < \Lambda_h(F) < +\infty$ and $\Lambda_h(E \cap B(x,r)) \leq 2h(r)$ for all $r \in (0,\delta)$ and all $x \in F$. Indeed, if such a subset does not exist, then, for Λ_h-almost $x \in E$, there exists a sequence of radius $r_n \to 0$ such that $\Lambda_h(E \cap B(x, r_n)) > 2h(r_n)$. By the Vitali covering lemma (see the previous section), we can cover Λ_h-almost all E by a disjoint union of such balls and thus we get the contradiction: $\Lambda_h(E) \geq 2\Lambda_h(E)$. We can assume that $\mathrm{diam} F \leq \delta$. Moreover, since $\int_0^\varepsilon h(r)^\alpha r^{-\alpha-1} dr < +\infty$ for any $\alpha > 1$, $\dfrac{h(r)}{r} \to 0$ and therefore $h(r) \leq Cr$ for all $0 < r \leq \delta$. Hence, for all (closed) ball B of $\mathbb{C}$, $\Lambda_h(F \cap B) \leq C \mathrm{diam} B$. We will see in the next chapter (see theorem 79) that the existence of a positive measure supported on F with linear growth and finite Menger curvature implies that $\gamma(F) > 0$, and thus $\gamma(E) > 0$.

PROOF. The classical idea to construct sets with projections of zero length is to use "Venetian blinds". See for instance the construction due to Besicovitch given in [**36**], pages 90-92. The construction of H. Joyce and P. Mörters follows the same strategy.

We first fix some parameters of the construction:

- Pick angles α_i with $\alpha_j = \dfrac{\pi}{2^n}$ if $2^n \leq j < 2^{n+1}$;
- Choose a constant $M > 3$ such that $M^{-1} < \dfrac{1}{12}(k+1)\sin(\alpha_{k+1})$ for any $k \geq 1$;
- Set $m_k = M.k$ and write $m(k) = m_1 \dots m_k$;
- Find an increasing sequence of numbers $\dfrac{1}{2} < \beta_k < 1$ converging to 1 so slowly that $(k+1)^2 m(k)^{\beta_{16m(k)}-1} \longrightarrow 0$ when $k \to +\infty$;
- Set $\sigma_k = \left(\dfrac{k+1}{k}\right)^{\beta_k}$.

First Step.
Let E_0 be a closed ball of diameter 1 in $\mathbb{C}$.

Second step.
Place m_1 balls of diameter $\dfrac{\sigma_1}{m_1}$ inside E_0 such that the centers are on the diameter of E_0 which makes an angle of α_1 with real axis. We place these balls such that they overlap as little as possible: two of these balls touch the boundary of E_0 and the

distance between the centers of two neighboring balls is $d_1 = \dfrac{1 - \frac{\sigma_1}{m_1}}{m_1 - 1}$.
These balls are called balls of the first generation and their union is denoted by E_1.

$(k+1)$-*th step.*

Assume now that E_k has been constructed. We put into every $m(k)$ balls of E_k exactly m_{k+1} closed balls. The centers are on a diameter making an angle of $\sum_{i=1}^{k+1} \alpha_i$ with the real axis, the diameter of these balls are $\dfrac{\sigma_1....\sigma_{k+1}}{m(k+1)}$ and the distance between the centers of two neighboring balls is equal to

$$d_{k+1} = \frac{\sigma_1....\sigma_k}{m_1....m_k} \frac{1 - \frac{\sigma_{k+1}}{m_{k+1}}}{m_{k+1} - 1}.$$

These balls are called balls of the $(k+1)$-th generation and their union is denoted by E_{k+1}.

Assume now that, by induction, the E_k's have been constructed. Set $E = \cap_{k=0}^{+\infty} E_k$. The crucial point is that two balls of the $(k+1)$-th generation intersect only if they are in the same ball of the generation k (by the choice of M). This allows us to code our set, that is there exists a natural bijection $p : \Pi_{k=1}^{+\infty}\{1, ..., m_k\} \to E$.
We now construct the function h. For this, define a piecewise linear function h by $h(r) = \dfrac{r}{\sigma_1....\sigma_k}$ if $d_{k+1} \le r < d_k$. Thus, since

$$\sum_{B \text{ balls of generation } k} h(\text{diam}B) < m(k) \frac{\frac{\sigma_1....\sigma_k}{m(k)}}{\sigma_1.....\sigma_k} < 1,$$

$\Lambda_h(E) < +\infty$. To see that $\Lambda_h(E) > 0$, consider the probability measure μ on E defined by $\mu(p((m_i)_{i=1,....,k})) = \dfrac{1}{m_1.....m_k}$. Then, by the choice of the parameters, $\mu(B(x,r)) \le Ch(r)$ for all $x \in E$ and all $r \in (0, \varepsilon)$. Cover now E by balls $B(x_i, r_i)$, $r_i \le \varepsilon$, such that $\sum_i h(r_i) \le 10\Lambda_h(E)$. Then, $\Lambda_h(E) \ge C \sum_i \mu(B(x_i, r_i)) \ge C\mu(E) > 0$.

Let $\alpha > 1$ and consider $I_\alpha = \displaystyle\int_0^\varepsilon h(r)^\alpha r^{-\alpha-1} dr$. First, note that

$$I_\alpha = \sum_{k=1}^{+\infty} \frac{1}{(\sigma_1....\sigma_k)^\alpha} \int_{d_{k+1}}^{d_k} \frac{dr}{r}.$$

Moreover, $\displaystyle\int_{d_{k+1}}^{d_k} \frac{dr}{r} = \log\left(\frac{d_k}{d_{k+1}}\right) \le C \log k$ for any $k \ge 1$. Furthermore, fix $\beta_n \ge \beta$ for some $\beta > \dfrac{1}{\alpha}$. Then, for $k \ge n$,

$$\frac{1}{(\sigma_1......\sigma_k)^\alpha} = \Pi_{j=1}^k \left(\frac{j}{j+1}\right)^{\beta_j \alpha} \le \left(\frac{1}{k}\right)^{\beta\alpha}$$

Finally, we get

$$I_\alpha \leq C \sum_{k=1}^{+\infty} \frac{\log k}{k^{\beta\alpha}} < \infty \text{ (Bertrand series).}$$

We now compute the length of the projection in the direction $\theta \in [0, \pi]$. For this, let $k_0 \in \mathbb{N}$ such that the diameter of a ball of the k_0-th generation is almost orthogonal to the direction θ. Let B a ball of generation k_0 and pick up a large $k \in \mathbb{N}$. Note that the distance from the center of a ball of generation $(k_0 + k)$ to the diameter of B (in direction θ) is bounded by

$$\text{diam} B \sin \left(\sum_{j=k_0+1}^{k_0+k} \alpha_j + \sum_{j=1}^{k_0} \alpha_j - \theta \right).$$

Hence, the projection of $B \cap E$ on L_θ is contained in an interval of length

$$|I_B| = 2 \left(\text{diam} B_{k_0+k} + \text{diam} B \left(\sin[\sum_{j=1}^{k_0+k} \alpha_j - \theta] \right) \right)$$

where B_{k+k_0} is a ball of generation $k + k_0$ in B.

Thus, we can cover the projection $p_\theta(E)$ of E by m_{k_0} intervals of length $|I_B|$. By the choice of the parameters, we can easily conclude. □

CHAPTER 7

The capacity γ_+ and the Painlevé Problem

In this chapter, we will see that γ and γ_+ are comparable. As an application, we will get a solution to the Painlevé Problem in terms of Menger curvature. All these results have been very recently obtained by J. Mateu, X. Tolsa and J. Verdera.

1. Melnikov's inequality

Let E be a subset of $\mathbb{C}$. We denote by $\mathcal{MC}(E)$ the class of positive Radon measures μ supported on E with linear growth with constant 1 (that is such that $\mu(B) \leq \mathrm{diam} B$ for any ball B in $\mathbb{C}$) and finite Menger curvature.

THEOREM 80. [**71**] *There exists $C_0 > 0$ such that for any compact set $E \subset \mathbb{C}$*

$$\gamma(E) \geq C_0 \sup_{\mu \in \mathcal{MC}(E)} \frac{\mu(E)^{\frac{3}{2}}}{(\mu(E) + c^2(\mu))^{\frac{1}{2}}}. \tag{82}$$

Note that if $\mathcal{MC}(E) = \emptyset$, then the supremum is 0 and (82) is obvious.

Remarks.
1) Let $E \subset \mathbb{R}^n$ and let $d > 0$. We define the d-Hausdorff content (denoted by $C^d(E)$) of E by

$$C^d(E) = \inf \left\{ \sum_i (\mathrm{diam} U_i)^d ; E \subset \bigcup_i U_i \right\}.$$

Theorem 31 and theorem 80 imply that

$$\gamma(E) \geq C \frac{C^1(E)^{\frac{3}{2}}}{l(\Gamma)^{\frac{1}{2}}}$$

whenever E is a compact subset of a rectifiable curve Γ of $\mathbb{C}$. This improves a previous result of T. Murai [**76**] and was observed by J. Verdera (private communications).
2) Let $E = E(\lambda)$ be a generalized four-corners Cantor set associated to a sequence λ (see chapter 3). Then, theorem 44 and theorem 80 imply that $\gamma(E) > 0$ if $\displaystyle\sum_n \left(\frac{4^{-n}}{\sigma_n} \right)^2 < +\infty$. We will see in the next section that the converse is also true.

PROOF. We follow here the proof given in [**105**] which is based on the $T1$ theorem for the Cauchy operator given in the previous chapter (whereas Melnikov's original proof uses the Garabedian formula).
Let $E \subset \mathbb{C}$ be a compact set. For any positive Radon measure μ supported on E, we define its "energy" (with respect to the Cauchy kernel $\frac{1}{z}$) by

$$\mathcal{E}(\mu) = \int_{\mathbb{C}} D_\mu(z) d\mu(z) + \int_{\mathbb{C}} c_\mu(z) d\mu(z)$$

H. Pajot: LNM 1799, pp. 105–114, 2002.

where $D_\mu(z) = \sup\limits_{r>0} \dfrac{\mu(D(z,r))}{r}$ and $c_\mu(z)^2 = \displaystyle\int_{\mathbb{C}}\int_{\mathbb{C}} c(x,y,z)^2 d\mu(x)d\mu(y)$.
We claim that theorem 80 follows from

PROPOSITION 81. *There exists $C > 0$ such that for each compact set $E \subset \mathbb{C}$,*

$$\gamma(E) \geq C \sup \mathcal{E}(\mu)^{-1}$$

where the supremum is taken over all probability measures μ supported on E.

Indeed, let μ be a positive Radon measure supported on E with linear growth and finite Menger curvature (that is $\mu \in \mathcal{MC}(E)$). Then, $D_\mu(z) \leq C\mu(\mathbb{C})$ since μ has linear growth and $\displaystyle\int_{\mathbb{C}} c_\mu d\mu(z) \leq Cc^2(\mu)\mu(\mathbb{C})^{\frac{1}{2}}$ by Cauchy-Schwarz. Thus,

$$\begin{aligned}
\mathcal{E}\left(\frac{\mu}{\mu(\mathbb{C})}\right) &= \mu(\mathbb{C})^{-2}\mathcal{E}(\mu) \\
&\leq C\mu(\mathbb{C})^{-2}\left(\mu(\mathbb{C}) + c^2(\mu)\mu(\mathbb{C})^{\frac{1}{2}}\right) \\
&\leq C\mu(\mathbb{C})^{-\frac{3}{2}}\left(\mu(\mathbb{C})^{\frac{1}{2}} + c^2(\mu)\right).
\end{aligned}$$

Hence, $\gamma(E) \geq C\dfrac{\mu(E)^{\frac{3}{2}}}{(\mu(E)+c^2(\mu))^{\frac{1}{2}}}$ and the proof of theorem 80 is complete.

We now prove proposition 81. For this, consider a probability measure μ supported on E such that $\mathcal{E}(\mu) < +\infty$. Set $A = 2\mathcal{E}(\mu)$ and $G = \{z \in E; D_\mu(z) \leq A$ and $c_\mu(z) \leq A\}$. Then,

$$A \geq \int_{\mathbb{C}\setminus G} D_\mu(z)d\mu(z) + \int_{\mathbb{C}\setminus G} c_\mu(z)d\mu(z) \geq 2A\mu(\mathbb{C}\setminus G).$$

Therefore, there exists a compact set $F \subset E$ such that $\mu(F) \geq \dfrac{1}{2}$ and $D_\mu(z) \leq A$, $c_\mu(z) \leq A$ whenever $z \in F$. Set $\nu = \mu_{\restriction F}$. Then,

(i) $\nu(\mathbb{C}) = \mu(F) \geq \dfrac{1}{2}$;
(ii) $\nu(D(z,R)) \leq AR$ whenever $z \in F$ and $R > 0$. Hence, $\nu(D(z,R)) \leq 4AR$ if $z \in \mathbb{C}$ and $R > 0$. Indeed, if $D(z,R) \cap F = \emptyset$, $\nu(D(z,R)) = 0$. Otherwise, there exists $\xi \in F \cap D(z,R)$. Therefore, $D(z,R) \subset D(\xi,4R)$ and $\nu(D(z,R)) \leq 4AR$.
(iii) $c_\nu(z) \leq A$ whenever $z \in F$.

Now, (ii) and (iii) imply

$$\int_D\int_D\int_D c(x,y,z)^2 d\nu(x)d\nu(y)d\nu(z) \leq C\nu(D)$$

for any disc D in $\mathbb{C}$. Therefore, by the proof of theorem 78 (see the comments after theorem 78), the Cauchy operator $\mathcal{C}_\nu$ is bounded on $L^2(\nu)$ and is of weak type $(1,1)$, that is there exists $C > 0$ such that

$$\nu\left(\{z \in \mathbb{C}, |\mathcal{C}_\nu^\varepsilon(\lambda)| < t\}\right) \leq \frac{CA}{t}\lambda(\mathbb{C})$$

where λ is any finite measure in $\mathbb{C}$ and $\varepsilon > 0$. Note the fact that the L^2 boundedness of the Cauchy operator $\mathcal{C}_\nu$ implies the weak L^1 inequality is classical if ν is doubling (see theorem 48). It is more delicate in the general case (see [**79**]). By the standard duality argument (theorem 71), there exists a measurable function h supported on

F such that $0 \leq h \leq 1$, $\nu(F) \leq 2\int_{\mathbb{C}} hd\nu$ and $|\mathcal{C}_\nu^\varepsilon(h)(z)| \leq CA$ if $z \in \mathbb{C} \setminus F$ and $\varepsilon > 0$. For the last estimate, recall that the constants C_1 and C_2 in theorem 71 are comparable. Therefore, $\gamma_+(F) \geq \frac{C}{A}\nu(F)$. Hence, since $\nu(F) > \frac{1}{2}$ and $A = 2\mathcal{E}(\mu)$,

$$\gamma(E) \geq C\frac{\nu(F)}{A} \geq C\mathcal{E}(\mu)^{-1}.$$

□

A natural question is to determine if the reverse estimate holds, that is

$$\gamma(E) \leq C \sup_{\mu \in \mathcal{MC}(E)} \frac{\mu(\mathbb{C})^{\frac{3}{2}}}{(\mu(\mathbb{C}) + c^2(\mu))^{\frac{1}{2}}}?$$

It was observed by X. Tolsa in his thesis that this estimate is true for the capacity γ_+. Recall that $\gamma_+(E) = \sup \mu(\mathbb{C})$ where the supremum is taken over all positive Radon measures μ supported on E such that $\left\|\frac{1}{z} * \mu\right\|_\infty \leq 1$.

Theorem 82. [**102**] *There exists $C > 0$ such that for any compact set $E \subset \mathbb{C}$,*

$$C^{-1} \sup_{\mu \in \mathcal{MC}(E)} \frac{\mu(\mathbb{C})^{\frac{3}{2}}}{(\mu(\mathbb{C}) + c^2(\mu))^{\frac{1}{2}}} \leq \gamma_+(E) \leq C \sup_{\mu \in \mathcal{MC}(E)} \frac{\mu(\mathbb{C})^{\frac{3}{2}}}{(\mu(\mathbb{C}) + c^2(\mu))^{\frac{1}{2}}}.$$

As easy applications, we get

Corollary 83. *Let $E \subset \mathbb{C}$ be a compact set. Then, $\gamma_+(E) = 0$ if and only if $\mathcal{MC}(E) = \emptyset$.*

Corollary 84 (Semi-additivity of γ_+). *There exists $C > 0$ such that*

$$\gamma_+(E_1 \cup E_2) \leq C(\gamma_+(E_1) + \gamma_+(E_2))$$

whenever E_1, E_2 are two disjoint compact sets in $\mathbb{C}$

To prove corollary 84, note that, if $\mu \in \mathcal{MC}(E_1 \cup E_2)$, then

- $\mu(E_1 \cup E_2) \geq \mu(E_1)$, $\mu(E_1 \cup E_2) \geq \mu(E_2)$ and $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$;
- $c^2(\mu) \geq c^2(\mu_{\restriction E_1}) + c^2(\mu_{\restriction E_2})$.

Thus,

$$\frac{\mu(\mathbb{C})^{\frac{3}{2}}}{(\mu(\mathbb{C}) + c^2(\mu))^{\frac{1}{2}}} \leq C\frac{\mu_1(\mathbb{C})^{\frac{3}{2}}}{(\mu_1(\mathbb{C}) + c^2(\mu_1))^{\frac{1}{2}}} + C\frac{\mu_2(\mathbb{C})^{\frac{3}{2}}}{(\mu_2(\mathbb{C}) + c^2(\mu_2))^{\frac{1}{2}}}$$

where $\mu_i = \mu_{\restriction E_i}$. By theorem 79, this implies $\gamma_+(E_1 \cup E_2) \leq C(\gamma_+(E_1) + \gamma_+(E_2))$.

Theorem 79 leads to the following question: Are γ and γ_+ comparable ? that is, does there exist $C > 0$ such that

$$\gamma_+(E) \leq \gamma(E) \leq C\gamma_+(E)$$

whenever $E \subset$ is a compact set ? Recall that the left hand side inequality is obvious (see chapter 5).

2. Tolsa's solution of the Painlevé problem

Very recently, X. Tolsa [**101**] proved that the answer to the previous question is yes.

THEOREM 85. *There exists $C > 0$ such that*

$$\gamma_+(E) \leq \gamma(E) \leq C\gamma_+(E)$$

whenever E is a compact set in $\mathbb{C}$. In particular, there exists $C > 0$ such that

$$C^{-1} \sup_{\mu \in \mathcal{MC}(E)} \frac{\mu(\mathbb{C})^{\frac{3}{2}}}{(\mu(\mathbb{C}) + c^2(\mu))^{\frac{1}{2}}} \leq \gamma(E) \leq C \sup_{\mu \in \mathcal{MC}(E)} \frac{\mu(\mathbb{C})^{\frac{3}{2}}}{(\mu(\mathbb{C}) + c^2(\mu))^{\frac{1}{2}}}.$$

Note that theorem 85 was first proved for generalized four-corners Cantor sets by J. Mateu, X. Tolsa and J. Verdera [**61**].The general proof follows the same way, but is more complicated and more technical. We will only describe their method in the case of the linear Cantor set (J. Verdera, personal communications). Before, we give some very important applications of theorem 85. We use the same notations as in chapter 3 (section 4).

THEOREM 86. [**61**] *Let $E = E(\lambda)$ be a generalized four-corners Cantor set associated to a sequence $\lambda = (\lambda_n)$. Then, for any $N \in \mathbb{N}$,*

$$C^{-1} \left(\sum_{j=1}^{N} \left(\frac{4^{-j}}{\sigma_j} \right)^2 \right)^{-\frac{1}{2}} \leq \gamma(E_N) \leq C \left(\sum_{j=1}^{N} \left(\frac{4^{-j}}{\sigma_j} \right)^2 \right)^{-\frac{1}{2}}.$$

By adapting Mattila's proof of theorem 44, we get $c^2(\mu_N) \leq C \sum_{j=1}^{N} \left(\frac{4^{-j}}{\sigma_j} \right)^2$ where μ_N is the restriction of Λ_h to E_N. Thus, the left hand side inequality follows from Melnikov's inequality and this bound of the curvature of μ_N. The right hand side inequality is a consequence of theorem 85 and a similar estimate for $\gamma_+(E_N)$ (see the work of Eidermann [**34**] or Tolsa [**102**])). In the linear case, the best known result was due to T. Murai (see [**75**]) who proved that $\gamma(E_N) \leq \dfrac{C}{\log N}$ (whereas theorem 86 gives $\gamma(E_N) \leq \dfrac{C}{\sqrt{N}}$).

COROLLARY 87. *Let $E = E(\lambda)$ be a generalized four-corners Cantor set associated to a sequence λ. Then, $\gamma(E) = 0$ (that is E is removable for bounded analytic functions) if and only if*

$$\left(\sum_{n} \left(\frac{4^{-n}}{\sigma_n} \right)^2 \right) = +\infty.$$

THEOREM 88 (Semi-additivity of the analytic capacity). *There exists $C > 0$ such that*

$$\gamma(E_1 \cup E_2) \leq C(\gamma(E_1) + \gamma(E_2))$$

whenever E_1, E_2 are two disjoint compact sets in $\mathbb{C}$

The proof is similar to those of corollary 84. In fact, Tolsa's proof gives a better result: There exists $C > 0$ such that

$$\gamma(\cup_{i \in \mathbb{N}} E_i) \leq C \sum_{i \in \mathbb{N}} \gamma(E_i)$$

whenever the E_i's are compact sets in $\mathbb{C}$. This suggests that the best constant in theorem 88 should be 1. Theorem 88 confirms an old conjecture of A. G. Vitushkin and has a lot of applications to uniform rational approximation (see [**62**]).

THEOREM 89. *Let $E \subset \mathbb{C}$ be a compact subset of $\mathbb{C}$. Then, $\gamma(E) > 0$ (that is E is not removable for bounded analytic functions) if and only if $\mathcal{MC}(E) \neq \emptyset$.*

Of course, theorem 89 provides a solution to the Painlevé problem (see the discussion in the next section).

Remark. Using theorem 89 and some estimates of the Menger curvature obtained in chapter 3, we can give alternative proofs of some results about removability for bounded analytic functions described in chapters 5 and 6. For instance, theorem 89 and corollary 32 imply theorem 69.

PROOF. We now prove theorem 85 in the special case of the N-th generation of the linear four-corners Cantor set (see chapter 1 section 3). We will use the same notations as above, that is E_N is the N-th generation of the four corners Cantor set and $E_N = \bigcup_{j=1}^{4^N} Q_N^j$ where Q_N^j is a square whose side length is 4^{-N}. We should mention that, throughout all the proof, we will use the notation $\mathcal{C}(\mu)$ (instead of $\mathcal{C}_\mu$) for the Cauchy transform of a measure μ. The proof is based on the following result.

LEMMA 90. *Assume that N is even (respectively odd) and that, for some constant $C_0 > 1$, we have $\gamma(E_{\frac{N}{2}}) \leq C_0\gamma(E_N)$ (respectively $\gamma(E_{\frac{N+1}{2}}) \leq C_0\gamma(E_N)$). Then, there exists an absolute constant $A > 0$ such that $\gamma(E_N) \leq C_0 A\gamma_+(E_{\frac{N}{2}})$ (respectively $\gamma(E_N) \leq C_0 A\gamma(E_{\frac{N+1}{2}})$).*

Assume for a moment that this lemma is proved. Let $N \in \mathbb{N}$ be even (the case where N is odd can be treated in a similar manner). Recall that $\gamma_+(E_n)$ is comparable to $\frac{1}{\sqrt{n}}$ for any $n \in \mathbb{N}$ (Eiderman-Tolsa's estimate, see above), and therefore there exists an absolute constant $C_1 > 0$ such that $\gamma_+(E_{\frac{N}{2}}) \leq C_1\gamma_+(E_N)$. We suppose by induction that $\gamma(E_m) \leq AC_1^2\gamma_+(E_m)$ for any $m < N$.

Case 1. $\gamma(E_N) \leq \frac{1}{C_1}\gamma(E_{\frac{N}{2}})$.

Then,

$$\begin{aligned}\gamma(E_N) &\leq \frac{1}{C_1}\gamma(E_{\frac{N}{2}})\\ &\leq AC_1\gamma_+(E_{\frac{N}{2}}) \text{ by induction hypothesis}\\ &\leq AC_1^2\gamma_+(E_N).\end{aligned}$$

Case 2. $\gamma(E_{\frac{N}{2}}) \leq C_1\gamma(E_N)$.

By lemma 90,

$$\begin{aligned}\gamma(E_N) &\leq AC_1\gamma_+(E_{\frac{N}{2}})\\ &\leq AC_1^2\gamma_+(E_N)\end{aligned}$$

We now go through the proof of lemma 90. For this, fix $N \in \mathbb{N}$ and assume that N is even. The main idea of the proof is to construct a measure λ supported on $\partial E_{\frac{N}{2}}$ such that

(i) λ is doubling with absolute constant;
(ii) $\lambda(\mathbb{C}) = \gamma(E_N)$;
(iii) The Cauchy operator $\mathcal{C}_\lambda$ is bounded on $L^2(\lambda)$ and its norm $||\mathcal{C}_\lambda||_{2,2}$ is bounded by $C\ C_0$ (for some absolute constant $C > 0$).

If such a measure has been constructed, $\mathcal{C}_\lambda$ is of weak type $(1,1)$ with constant bounded by $C\ C_0$ and therefore, by the standard duality argument (theorem 71), there exists h supported on $\partial E_{\frac{N}{2}}$ such that $0 \leq h \leq 1$, $\lambda(E_{\partial\frac{N}{2}}) \leq \sup_\mu 2\int_{\mathbb{C}} hd\mu$ and $|\mathcal{C}(hd\lambda)(z)| \leq C\ C_0$ for $z \notin \partial E_{\frac{N}{2}}$. For the last estimate, recall that constants C_1 and C_2 in theorem 71 are comparable. Hence,there exists $A > 0$ such that $\gamma(E_N) = \lambda(\mathbb{C}) \leq A\ C_0 \gamma_+(E_{\frac{N}{2}})$ and the lemma 90 is proved.

Construction of λ

Let f be the Ahlfors function of E_N (see chapter 5). Then, for $z \notin E_N$,

$$f(z) = \frac{-1}{2i\pi}\int_{\partial E_N} \frac{f(s)t(s)}{s-z} ds$$

where $t(s)$ is the unit tangent vector. Set $\nu = \dfrac{-1}{2i\pi} f(s)t(s)ds_{\restriction \partial E_N}$. Then,

- ν is supported on ∂E_N;
- $|\mathcal{C}(\nu)(z)| \leq 1$ if $z \notin E_N$;
- If $Q = Q_n^j$, $0 \leq n \leq N$, $1 \leq j \leq 4^n$, set $f_Q(z) = \dfrac{-1}{2i\pi}\displaystyle\int_{\partial E_N \cap Q} \frac{f(s)t(s)}{s-z} ds$ for $z \notin \partial E_N \cap Q$. Then, f_Q is analytic outside $Q \cap E_N$. Moreover, f_Q is bounded (uniformly in Q) outside $Q \cap E_N$. Indeed, let $z \in \mathbb{C} \setminus (Q \cap E_N)$. Assume first that $d(z, E_N \cap Q) \geq \varepsilon_0 \mathrm{diam} Q$ (where ε_0 is small enough). Then,

$$|f_Q(z)| \leq \frac{||f||_\infty H^1(Q \cap \partial E_N)}{2\pi\varepsilon_0\ \mathrm{diam}Q} \leq C \tag{83}$$

Assume now that $d(z, Q \cap \partial E_N) \leq \varepsilon_0\ \mathrm{diam}Q$ and consider $\tilde{Q}$ a square concentric with Q but whose side length is 100 diam$Q$. Then, $f(z) - f_Q(z) = \dfrac{1}{2i\pi}\displaystyle\int_{\partial\tilde{Q}} \frac{f(w)}{w-z} dH^1(w)$. An estimate similar to (83) gives

$$\left| \frac{1}{2i\pi}\int_{\partial\tilde{Q}} \frac{f(w)}{w-z} dw \right| \leq C.$$

Thus, $|f_Q(z)| \leq C+1$. Therefore, $|f'_Q(\infty)| = |\nu(Q)| \leq C\gamma(Q \cap E_N)$ and then, by translating and dilating, $|\nu(Q)| \leq C4^{-n}\gamma(E_{N-n})$. In particular, if $n = \dfrac{N}{2}$, we have

$$\begin{aligned} |\nu(Q)| &\leq C4^{-\frac{N}{2}}\gamma(E_{\frac{N}{2}}) \\ &\leq CC_0\gamma(E_N)\mu(Q) \text{ by hypothesis} \end{aligned}$$

(where μ is the arc length measure on $\partial E_{\frac{N}{2}}$).

Set $\lambda = \dfrac{1}{4}\gamma(E_N)\mu = \dfrac{1}{4}\gamma(E_N)ds_{\restriction \partial E_{\frac{N}{2}}}$. Then, λ is a positive measure supported on $\partial E_{\frac{N}{2}}$ with $\lambda(\mathbb{C}) = \gamma(E_N)$ and satisfies for $0 \leq n \leq \dfrac{N}{2}$ and $1 \leq j \leq 4^n$,

$$|\nu(Q_n^j)| \leq CC_0\lambda(Q_n^j). \tag{84}$$

$\mathcal{C}_\lambda$ is bounded on $L^2(\lambda)$

Note first that λ is doubling with absolute constant (λ is also Ahlfors regular, but with constant depending on $\gamma(E_N)$). To prove that the Cauchy operator $\mathcal{C}_\lambda$ is bounded on $L^2(\lambda)$, we will check that the hypothesis of M. Christ local $T(b)$ theorem (theorem 51) are satisfied with absolute constants, that is for any Q_n^j, $0 \leq n \leq \dfrac{N}{2}$, $1 \leq j \leq 4^n$, there exists b_j^n supported on Q_n^j such that

(i) $||b_j^n||_\infty \leq CC_0$;
(ii) $||\mathcal{C}(b_j^n d\lambda)||_\infty \leq C$;
(iii) $\left|\int_{Q_n^j} b_j^n d\lambda\right| \geq \delta\lambda(Q_n^j)$

where C and δ are absolute constants. Furthermore, the proof of theorem 51 implies that $||\mathcal{C}_\lambda||_{2,2} \leq CC_0$ for some absolute constant $C > 0$

Construction of the b_j^n's

Let Q^0 be the initial square. Recall that $\mathcal{C}(\nu)$ is bounded. We would like to write $\nu = bd\lambda$ and set $b^0 = b$ (where b^0 is the function associated to Q^0). But, ν is absolutely continuous with respect to $ds_\restriction E_N$ but not with respect to $ds_{\restriction E_{\frac{N}{2}}}$. Thus, we should adapt ν to $\partial E_{\frac{N}{2}}$. To do this, set

$$b = \sum_{j=1}^{4^{\frac{N}{2}}} \frac{\nu(Q_{\frac{N}{2}}^j)}{\lambda(Q_{\frac{N}{2}}^j)}\chi_{Q_{\frac{N}{2}}^j}.$$

Then,

(i) $||b||_\infty \leq CC_0$ since $|\nu(Q_{\frac{N}{2}}^j)| \leq CC_0\lambda(Q_{\frac{N}{2}}^j)$;
(ii) $||C(bd\lambda)||_\infty \leq C$. To see this, the idea now is to compare $C(bd\lambda)$ with $C(\nu)$. Assume that z is closed to $\partial E_{\frac{N}{2}}$, that is $z \in 2Q_{\frac{N}{2}}^{j_0}$ for some j_0. Write $bd\lambda = \sum_{j=1}^{4^{\frac{N}{2}}} \nu(Q_{\frac{N}{2}}^j)\dfrac{ds_{\restriction \partial Q_{\frac{N}{2}}^j}}{l(\partial Q_{\frac{N}{2}}^j)}$ and $\nu = \sum_{j=1}^{4^{\frac{N}{2}}} \chi_{Q_{\frac{N}{2}}^j}\nu$. Then, using the fact that

$$|\nu(Q_{\frac{N}{2}}^j)| = |(bd\lambda)(Q_{\frac{N}{2}}^j)|,$$

we get

$$\begin{aligned}\left|\mathcal{C}\left(\sum_{j\neq j_0}\nu(Q_{\frac{N}{2}}^j)\frac{ds_{\restriction \partial Q_{\frac{N}{2}}^j}}{l(\partial Q_{\frac{N}{2}}^j)} - \sum_{j\neq j_0}\chi_{Q_{\frac{N}{2}}^j}\nu\right)(z)\right| &\leq C\sum_{j\neq j_0}\frac{l(\partial Q_{\frac{N}{2}}^j)^2}{d(z, Q_{\frac{N}{2}}^j)^2} \\ &\leq C.\end{aligned}$$

Since $\mathcal{C}(\nu)$ is bounded, we are done if $\mathcal{C}(ds_{\restriction \partial Q_{\frac{N}{2}}^{j_0}})(z)$ is bounded. If $Q_{\frac{N}{2}}^{j_0}$ is a disc, it is obvious. Thus, up to some technical problems (smoothing the boundary of $Q_{\frac{N}{2}}^{j_0}$ since the problem comes from the corners of $Q_{\frac{N}{2}}^{j_0}$), we can assume that it is true. We leave to the reader the case where z is far away from $\partial E_{\frac{N}{2}}$.

(iii) $\displaystyle\int_{Q^0} b d\lambda = \sum_{j=1}^{4^{\frac{N}{2}}} \nu(Q^j_{\frac{N}{2}}) = \nu(\mathbb{C}) = \gamma(E_N) = \lambda(\mathbb{C}) = \lambda(Q^0).$

Fix now Q^j_n, $0 \leq n \leq \dfrac{N}{2}$, $0 \leq j \leq 4^n$. As before, comparing with ν, we prove that the measure $m = \displaystyle\sum_{k=1}^{4^{\frac{N}{2}-n}} \nu(Q^k_{\frac{N}{2}-n}) \frac{ds_{\restriction Q^k_{\frac{N}{2}-n}}}{l(\partial Q^k_{\frac{N}{2}-n})}$ has bounded Cauchy transform. The idea now is to "transport" the measure m. To do this, write $Q^j_n = 4^{-n}Q^0 + z^n_j$ and set $\tau : z \to \tau(z) = 4^{-n}z + z^n_j$. We can assume that $Q^k_{\frac{N}{2}} = \tau(Q^k_{\frac{N}{2}-n})$ for any $k = 1, ..., 4^{\frac{N}{2}-n}$. Then, the image measure τ_m of m by τ is given by $\tau_m = \displaystyle\sum_{k=1}^{4^{\frac{N}{2}-n}} \nu(Q^k_{\frac{N}{2}-n}) \frac{ds_{\restriction \partial Q^k_{\frac{N}{2}}}}{l(\partial Q^k_{\frac{N}{2}})}$. Since $\mathcal{C}(\tau_m)(z) = 4^n \mathcal{C}(m)(4^n(z - z^n_j))$, $4^{-n}\tau_m$ has bounded Cauchy transform. Now, consider b^n_j supported on Q^j_n such that

$$4^{-n}\tau_m = \sum_{k=1}^{4^{\frac{N}{2}-n}} \frac{\nu(Q^k_{\frac{N}{2}-n})}{\lambda(Q^k_{\frac{N}{2}-n})} \chi_{Q^k_{\frac{N}{2}}} d\lambda = b^n_j d\lambda.$$

Then, b^n_j satisfies

(i) $||b^n_j||_\infty \leq CC_0$ since $\nu(Q^k_{\frac{N}{2}-n}) \leq CC_0 \lambda(Q^k_{\frac{N}{2}-n})$;

(ii) $||\mathcal{C}(b^n_j d\lambda)||_\infty \leq C$ (see above);

(iii) $\displaystyle\int b^n_j d\lambda = 4^{-n}\tau_m(\mathbb{C}) = 4^{-n}\nu(\mathbb{C}) = 4^{-n}\gamma(E_N) = \lambda(Q^j_n).$

This completes the proof of lemma 90, and therefore of theorem 85. See [**62**] for another presentation of this proof and for a discussion of the general case.

□

3. Concluding remarks and open problems

Does theorem 85 provide a satisfactory solution to the Painlevé problem (for instance, in the spirit of L. Ahlfors' quotation given in the introduction) ?

We can not expect to give a definitive answer to this question. However, we will describe two open problems related to removability for bounded analytic functions. Question 1 (respectively question 3) has a metric (respectively geometric) nature. Using Tolsa's result, they can be restated in terms of Menger curvature. Therefore, in some sense, answering to question 1 and question 3 by the use of theorem 85 would mean that Tolsa's result provides a metric/geometric characterization of removable sets for bounded analytic functions.

On the other hand, as we have seen in chapter 4, it is quite hard to estimate the Menger curvature, and therefore to construct measures with finite curvature supported on a given set. In other words, in order to prove that a set is removable for bounded analytic functions, it is not so easy to check hypothesis of theorem 85, or just to have the feeling that a measure with finite curvature and supported on the set exists ! Hence, it would be useful to find (at least in some special situations) more "visual" conditions.

Question 1: Is the class of removable sets for bounded analytic functions stable under bilipschitz homeomorphism $f : \mathbb{C} \to \mathbb{C}$?

Recall that $f : \mathbb{C} \to \mathbb{C}$ is bilipschitz if there exists $C \geq 1$ such that

$$C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

whenever x, $y \in \mathbb{C}$.
An answer "yes" to the following question will imply an answer "yes" to question 1.

Question 2: Let μ be a positive Radon measure in $\mathbb{C}$ with linear growth and finite Menger curvature and let $f : \mathbb{C} \to \mathbb{C}$ be a bilipschitz homeomorphism. Denote by μ_f the image measure of μ by f. Is the Menger curvature of μ_f finite ?
Note that μ_f has linear growth. The answer is "yes" if f is affine or of class $C^{1+\varepsilon}$ (see [**99**]). Therefore, the class of removable sets is stable under affine functions. This gives an affirmative answer to a problem raised by A. O'Farrel who asked if $\gamma(f(E)) = 0$ whenever $\gamma(E) = 0$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the map given by $f(x, y) = (x, 2y)$ (see [**104**]). We can not expect to control $c^2(\mu_f)$ by $c^2(\mu)$. To see this, consider for instance $\mu = \mathcal{L}^1_{\lceil [0,1]}$. Then, for any bilipschitz homeomorphism $f : \mathbb{C} \to \mathbb{C}$, μ_f is supported by a chord-arc curve (which is a typical example of Ahlfors-regular curve) and in general, $c^2(\mu_f) > 0$ whereas $c^2(\mu) = 0$. A reasonable conjecture is that there exists $C > 0$ (depending on the bilipschitz constant of f and on the growth of μ) such that

$$c^2(\mu_f) \leq C(\mu(\mathbb{C}) + c^2(\mu)).$$

This estimate holds if μ is Ahlfors-regular [**88**]. See also [**45**] for some results in the case of Cantor type sets.

Question 3: Let E be a compact set such that $\mathrm{Fav}(E) > 0$. Is it true that E is not removable for bounded analytic functions ?
This questions is the open direction in the Vitushkin conjecture (see chapter 6). By theorem 85, it is equivalent to

Question 4: Let $E \subset \mathbb{C}$ be a compact set such that $\mathrm{Fav}(E) > 0$. Does there exist a positive Radon measure μ supported on E with linear growth and finite Menger curvature (that is $\mathcal{MC}(E) \neq \emptyset$) ?

A typical example of compact sets E in $\mathbb{C}$ with $\mathrm{Fav}(E) > 0$ is a continuum. In this case, such a measure have been constructed in chapter 4. It is not clear how to adapt these arguments in the general case.

Note that the relationship between analytic capacity and Favard length is far away to be understood. As far as I know, the only result is due to T. Murai [**76**]: If E is a subset of a Lipschitz graph Γ, then $\gamma(E) \geq C\dfrac{\mathrm{Fav}(E)^{\frac{3}{2}}}{l(\Gamma)^{\frac{1}{2}}}$. However, if E_N is the N-th generation of the linear 4-corners Cantor set, we know that $\gamma(E_N)$ is comparable to $\dfrac{1}{\sqrt{N}}$. On the other hand, it is known that $\mathrm{Fav}(E_N) \geq \dfrac{C^{-1}}{N}$ [**64**]. A reasonable conjecture is that $\mathrm{Fav}(E_N) \leq \dfrac{C}{N}$ and thus $\mathrm{Fav}(E_N)$ is comparable to $\gamma(E_N)^2$. An upper bound is given in [**89**] (where the interested reader can find more details about this problem), but this bound is much bigger that $\frac{C}{N}$.

A more challenging problem is to extend most of the results of the last two chapters in higher dimensions, that is in the setting of the study of removable sets for harmonic Lipschitz functions in $\mathbb{R}^n$. As noted previously, the main obstacle to do this is the lack of an appropriate "Menger curvature". Recall that a compact set $E \subset \mathbb{R}^n$ is removable for Lipschitz harmonic functions if for any open set U containing E and any Lipschitz function $f : U \to \mathbb{R}$ which is harmonic in $U \setminus E$, f is harmonic in U. If $n = 2$ and if $H^1(E) < +\infty$, then E is removable for bounded analytic functions if and only if E is removable for Lipschitz harmonic functions. This follows from the results of [**26**] and [**27**]. See [**65**] and [**70**] for the state of the art concerning this problem.

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